Problem 12011
(American Mathematical Monthly, Vol.124, December 2017)

Proposed by C. I. Valean (Romania).

Calculate

$$\lim_{n\to\infty}\left(\frac{1}{n!}\int_0^\infty\int_0^\infty \frac{x^n-y^n}{e^x-e^y}\,dx\,dy-2n\right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We have that

$$\int_0^\infty\int_0^\infty \frac{x^n-y^n}{e^x-e^y}\,dx\,dy=2\int_0^\infty e^{-x}dx\int_{y=0}^x \frac{x^n-y^n}{1-e^{-(x-y)}}\,dy=2A_n+2B_n,$$

where

$$\begin{aligned}A_n&=\int_0^\infty e^{-x}dx\int_{z=0}^x (x^n-y^n)\,dy\\&=\int_0^\infty x^{n+1}e^{-x}dx-\frac{1}{n+1}\int_0^\infty x^{n+1}e^{-x}dx=(n+1)!-n!=n\,n!\end{aligned}$$

and

$$\begin{aligned}B_n&=\int_0^\infty e^{-x}dx\int_{y=0}^x \frac{(x^n-y^n)e^{-(x-y)}}{1-e^{-(x-y)}}\,dy\\
&=\int_0^\infty e^{-x}dx\int_{y=0}^x (x^n-y^n)\sum_{j=1}^\infty e^{-j(x-y)}\,dy\\
&=\sum_{j=1}^\infty\left(\int_0^\infty x^ne^{-(j+1)x}dx\int_{y=0}^x e^{jy}\,dy-\int_0^\infty e^{-(j+1)x}dx\int_{y=0}^x y^ne^{jy}\,dy\right)\\
&=\sum_{j=1}^\infty\left(\frac{1}{j}\int_0^\infty x^ne^{-(j+1)x}\left(e^{jx}-1\right)dx-\frac{(-1)^nn!}{j^{n+1}}\int_0^\infty e^{-(j+1)x}\left(e^{jx}\sum_{k=0}^n\frac{(-jx)^k}{k!}-1\right)dx\right)\\
&=\sum_{j=1}^\infty\left(\frac{1}{j}\int_0^\infty x^ne^{-x}dx-\frac{1}{j}\int_0^\infty x^ne^{-(j+1)x}dx-\frac{(-1)^nn!}{j^{n+1}}\left(\sum_{k=0}^n\frac{(-j)^k}{k!}\int_0^\infty x^ke^{-x}dx-\int_0^\infty e^{-(j+1)x}dx\right)\right)\\
&=n!\sum_{j=1}^\infty\left(\frac{1}{j}-\frac{1}{j(j+1)^{n+1}}-\frac{(-1)^n}{j^{n+1}}\sum_{k=0}^n(-j)^k+\frac{(-1)^n}{j^{n+1}(j+1)}\right)\\
&=n!\sum_{j=1}^\infty\left(\frac{1}{j}-\frac{1}{j(j+1)^{n+1}}-\frac{(-1)^n}{j^{n+1}}\cdot\frac{1-(-j)^{n+1}}{1-(-j)}+\frac{(-1)^n}{j^{n+1}(j+1)}\right)\\
&=n!\sum_{j=1}^\infty\left(\frac{1}{j}-\frac{1}{j(j+1)^{n+1}}-\frac{1}{j+1}\right)=n!\left(1-\sum_{j=1}^\infty\frac{1}{j(j+1)^{n+1}}\right).\end{aligned}$$

Hence, as n goes to infinity,

$$\frac{1}{n!}\int_0^\infty\int_0^\infty \frac{x^n-y^n}{e^x-e^y}\,dx\,dy-2n=\frac{2A_n+2B_n}{n!}-2n=2-2\sum_{j=1}^\infty\frac{1}{j(j+1)^{n+1}}\to 2$$

because

$$0\le\sum_{j=1}^\infty\frac{1}{j(j+1)^{n+1}}\le\frac{1}{2^n}\sum_{j=1}^\infty\frac{1}{j(j+1)}=\frac{1}{2^n}\to 0.$$

$\square$