Problem 12006
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When n is an integer and $n \geq 2$, let $a_n = \lceil n/\pi \rceil$ and $b_n = \lceil \csc(\pi/n) \rceil$. The sequences $a_2, a_3, \ldots$ and $b_2, b_3, \ldots$ are, respectively,

$$1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 7, 7, 7, 8, 8, 8, 8, 9, \ldots$$

and

$$1, 2, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 7, 7, 7, 8, 8, 8, 8, 9, \ldots$$

They differ when $n = 3$. Are they equal for all larger n?

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. NO, they are not equal for all $n > 3$.
Since $\sin(x) < x$ for $x > 0$, then $b_n \geq a_n$. For $n \geq 2$, the strict inequality $b_n > a_n$ holds if there is $m \in \mathbb{N}^+$ such that $n/\pi < m < \csc(\pi/n)$ that is

$$n \sin(\pi/n) < \frac{n}{m} < \pi.$$

Since $\sin(x) < x - \frac{x^3}{7}$ for $x \in (0, \pi/2]$, it suffices that

$$\pi - \frac{\pi^3}{7n^2} < \frac{n}{m} < \pi \quad \Leftrightarrow \quad 0 < \pi - \frac{n}{m} < \frac{\pi^3}{7n^2},$$

which is implied by (note that $\pi^2 > n^2/m^2$ and $\pi/7 > 1/\sqrt{5}$)

$$0 < \pi - \frac{n}{m} < \frac{1}{\sqrt{5}\, m^2}.$$

Hence, by Hurwitz's theorem, the above inequality is satisfied by *good* lower Diophantine approximations of the irrational number π. For example, by using the continued fraction convergents of π, we have that the inequality holds for $n = 3$ and $m = 1$ when

$$a_3 = 1 < 2 = b_3,$$

but also for $n = 80143857$ and $m = 25510582$ when

$$a_{80143857} = 25510582 < 25510583 = b_{80143857}.$$

□