Problem 12004
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Proposed by M. Omarjee (France).

Let $\{a_n\}_{n\geq 1}$ *be a strictly increasing sequence of real numbers such that* $a_n \leq n^2\ln(n)$ *for all* $n \geq 1$. *Prove that the series*

$$\sum_{n=1}^{\infty} \frac{1}{a_{n+1}-a_n}$$

diverges.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Assume on the contrary that the series (with positive terms) is convergent.
Then (note that $a_1 \leq 1^2\ln(1) \leq 0$)

$$\sum_{n=1}^{\infty} \frac{1}{a_{n+1}-a_n} = \sum_{k=1}^{\infty}\left(\sum_{n=2^{k-1}}^{2^k-1} \frac{1}{a_{n+1}-a_n}\right) < +\infty = \sum_{k=1}^{\infty} \frac{1}{4(\ln(2)k-\frac{a_1}{4^k})}$$

which implies that there exists a positive integer k such that

$$\sum_{n=2^{k-1}}^{2^k-1} \frac{1}{a_{n+1}-a_n} < \frac{1}{4(\ln(2)k-\frac{a_1}{4^k})}.$$

Moreover, by Cauchy-Schwarz inequality,

$$\left(\sum_{n=2^{k-1}}^{2^k-1} \frac{1}{a_{n+1}-a_n}\right)\cdot\left(\sum_{n=2^{k-1}}^{2^k-1}(a_{n+1}-a_n)\right) \geq \left(\sum_{n=2^{k-1}}^{2^k-1} 1\right)^2 = (2^k-2^{k-1})^2 = 4^{k-1}.$$

Therefore

$$\frac{1}{4(\ln(2)k-\frac{a_1}{4^k})} > \sum_{n=2^{k-1}}^{2^k-1} \frac{1}{a_{n+1}-a_n} \geq \frac{4^{k-1}}{\sum_{n=2^{k-1}}^{2^k-1}(a_{n+1}-a_n)} = \frac{4^{k-1}}{a_{2^k}-a_{2^{k-1}}} \geq \frac{4^{k-1}}{a_{2^k}-a_1},$$

that is

$$a_{2^k}-a_1 > 4^k\left(\ln(2)k-\frac{a_1}{4^k}\right) = 4^k\ln(2^k)-a_1 \implies a_{2^k} > (2^k)^2\ln(2^k)$$

which contradicts the given inequality $a_n \leq n^2\ln(n)$ for $n = 2^k$. □