Problem 12000
(American Mathematical Monthly, Vol.124, October 2017)

Proposed by M. Sawhney (USA).

Let $H_k = \sum_{i=1}^{k} 1/i$*. Prove that the function* $f : \mathbb{R} \to \mathbb{R}$ *defined by*

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{\prod_{k=1}^{n} H_k}$$

has no real zeroes.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Note that the radius of convergence of the power series $f(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$ is

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} H_{n+1} = +\infty.$$

Therefore f is well-defined and continuous in $\mathbb{R}$. Assume that f has at least a real zero and let

$$t := \sup\{x \in \mathbb{R} \ : \ f(x) = 0\}.$$

Note that $t < 0$ because $f(x) \geq 1$ for $x \geq 0$. Moreover, by continuity, $f(t) = 0$ and $f(x) > 0$ for $x \in (t, 0]$. Hence

$$I := \int_t^0 \frac{f(x) - f(t)}{x - t}\, dx = \int_t^0 \frac{f(x)}{x - t}\, dx > 0.$$

On the other hand, since power series can be integrated term-by-term on an interval lying inside the interval of convergence, we have that

$$\begin{aligned}
I &= \int_t^0 \sum_{n=1}^{\infty} \frac{x^n - t^n}{(x-t)\prod_{k=1}^{n} H_k}\, dx \\
&= \sum_{n=1}^{\infty} \frac{1}{\prod_{k=1}^{n} H_k} \int_t^0 \frac{x^n - t^n}{x - t}\, dx \quad (s := x/t) \\
&= -\sum_{n=1}^{\infty} \frac{t^n}{\prod_{k=1}^{n} H_k} \int_0^1 \frac{s^n - 1}{s - 1}\, ds \\
&= -\sum_{n=1}^{\infty} \frac{t^n}{\prod_{k=1}^{n} H_k} \sum_{k=0}^{n-1} \int_0^1 s^k\, ds \\
&= -\sum_{n=1}^{\infty} \frac{t^n H_n}{\prod_{k=1}^{n} H_k} \\
&= -t \left(1 + \sum_{n=2}^{\infty} \frac{t^{n-1}}{\prod_{k=1}^{n-1} H_k} \right) = -t f(t) = 0
\end{aligned}$$

which contradicts the fact that $I > 0$. So we may conclude that f has no real zeros. □