Problem 11999
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Proposed by O. Kouba (Syria).

Evaluate

$$\sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \sqrt{k}+\sqrt{k+1} \rfloor}}{k(k+1)}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let k be a positive integer. Since $4k+3$ and $4k+2$ are never squares, there exists a positive integer m such that

$$m^2 \leq \sqrt{4k+1} < \sqrt{4k+2} < \sqrt{4k+3} < (m+1)^2.$$

Moreover, it is easy to verify that for all $x \geq 0$,

$$\sqrt{4x+1} \leq \sqrt{x} + \sqrt{x+1} < \sqrt{4x+3}$$

and we may conlude that

$$\lfloor \sqrt{k} + \sqrt{k+1} \rfloor = \lfloor \sqrt{4k+1} \rfloor.$$

Now $\lfloor \sqrt{4k+1} \rfloor$ is equal to the even number $2n$ if and only if

$$(2n)^2 \leq 4k+1 < (2n+1)^2 \;\Leftrightarrow\; k \in [n^2, n^2+n).$$

Since the given series is absolutely convergent, we can rearrange the terms and, by noting that the finite sums are telescopic, we obtain

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \sqrt{k}+\sqrt{k+1} \rfloor}}{k(k+1)} &= \sum_{n=1}^{\infty} \left(\sum_{k=n^2}^{n^2+n-1} \left(\frac{1}{k} - \frac{1}{k+1} \right) - \sum_{k=n^2+n}^{(n+1)^2-1} \left(\frac{1}{k} - \frac{1}{k+1} \right) \right) \\
&= \sum_{n=1}^{\infty} \left(\left(\frac{1}{n^2} - \frac{1}{n^2+n} \right) - \left(\frac{1}{n^2+n} - \frac{1}{(n+1)^2} \right) \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} - 2 \sum_{n=1}^{\infty} \frac{1}{n^2+n} \\
&= 2 \sum_{n=1}^{\infty} \frac{1}{n^2} - 1 - 2 \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\
&= \frac{\pi^2}{3} - 3.
\end{aligned}$$

□