Problem 11996
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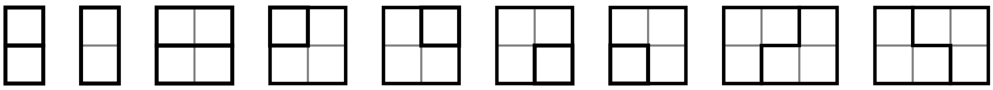
Proposed by R. Tauraso (Italy).

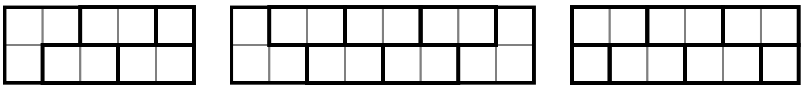
Consider all the tilings of a 2-by-n rectangle comprised of tiles that are either a unit square, a domino, or a right tromino. Let f_n be the fraction of tiles among all such tilings that are unit squares. For example, $f_2 = 4/7$, because 16 out of the 28 tiles in the 11 tilings of a 2-by-2 rectangle are squares. What is $\lim_{n\to+\infty} f_n$?

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We say that a tiling of the $(2 \times n)$-board is *unbreakable*, if it cannot be split vertically, i. e. if it can not be separated into a tiling of a $(2 \times k)$-board and a tiling of a $(2 \times (n-k))$-board for any $0 < k < n$. The unbreakable tilings are



together with the tilings which consist of a non-empty sequence of staggered horizontal dominoes with a square or a tromino on the left and with a square or a tromino on the right. For example:



Therefore the generating function of the number of unbreakable tilings is

$$U_{(s,d,t)}(x) = (s^2+d)x + (d^2+4st)x^2 + 2t^2x^3 + (s+tx)\left(2\sum_{k=2}^{\infty} d^{k-1}x^k\right)(s+tx)$$
$$= (s^2+d)x + (d^2+4st+2s^2d)x^2 + \frac{2(sd+t)^2x^3}{1-dx}$$

where the powers of the variables s, d, and t count the number of squares, dominoes, and right trominoes respectively. Therefore the generating function of the number of tilings is

$$F_{(s,d,t)}(x) = \frac{1}{1-U_{(s,d,t)}(x)} = \frac{(1-dx)}{1-(s^2+2d)x-(4st+s^2d)x^2-(2t^2-d^3)x^3}.$$

The number of squares s_n in all the tilings of the $(2 \times n)$-board is

$$s_n = [x^n]\left(\frac{d}{ds}\left(F_{(s,1,1)}(x)\right)\right)_{s=1} = [x^n]\frac{x(2+4x-6x^2)}{(1+x)^2(1-4x-x^2)^2}$$
$$= \frac{1}{2}\left(nF_{3n+1} + F_{3n-1} + (n-1)(-1)^n\right) \sim \frac{nF_{3n+1}}{2},$$

and the total number of pieces p_n is

$$p_n = [x^n]\left(\frac{d}{dp}\left(F_{(p,p,p)}(x)\right)\right)_{p=1} = [x^n]\frac{x(3+10x-5x^2)}{(1+x)^2(1-4x-x^2)^2}$$
$$= \frac{1}{20}\left((17n+10)F_{3n+1} + (4n-12)F_{3n} + (15n-10)(-1)^n\right) \sim \frac{n(17F_{3n+1}+4F_{3n})}{20}$$

where F_n denotes the n-th Fibonacci number. Hence the required limit is

$$\lim_{n\to+\infty} f_n = \lim_{n\to+\infty}\frac{s_n}{p_n} = \lim_{n\to+\infty}\frac{10}{17+4F_{3n}/F_{3n+1}} = \frac{10}{17+2(\sqrt{5}-1)} = \frac{10}{15+2\sqrt{5}} = \frac{30-4\sqrt{5}}{41}$$

which is approximately equal to 0.51355434. $\square$