Problem 11995
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Proposed by D. S. Marinescu and M. Monea (Romania).

Suppose $0 < x_0 < \pi$, *and for* $n \geq 1$ *define* $x_n = \frac{1}{n}\sum_{k=0}^{n-1}\sin(x_k)$. *Find* $\lim_{n\to\infty} x_n\sqrt{\ln(n)}$.

Solution proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France, and Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", Italy.

Solution. By the recurrence relation

$$x_{n+1} = \frac{\sum_{k=0}^{n-1}\sin(x_k) + \sin(x_n)}{n+1} = \frac{nx_n + \sin(x_n)}{n+1}$$

which implies that for $0 < x_0 < \pi$,

$$0 < x_{n+1} < \frac{nx_n + x_n}{n+1} = x_n$$

that is the sequence $(x_n)_n$ is positive and decreasing and therefore it has a limit l which satisfies $(n+1)l = nl + \sin(l)$. Thus it follows that $l = 0$. Moreover, by Taylor approximation,

$$x_{n+1} = \frac{nx_n + x_n - \frac{x_n^3}{6} + o(x_n^3)}{n+1} = x_n\left(1 - \frac{x_n^2}{6(n+1)} + \frac{o(x_n^2)}{n+1}\right)$$

and

$$\frac{1}{x_{n+1}^2} = \frac{1}{x_n^2}\left(1 - \frac{x_n^2}{6(n+1)} + \frac{o(x_n^2)}{n+1}\right)^{-2} = \frac{1}{x_n^2}\left(1 + \frac{x_n^2}{3(n+1)} + \frac{o(x_n^2)}{n+1}\right) = \frac{1}{x_n^2} + \frac{1}{3(n+1)} + \frac{o(1)}{n+1}.$$

Finally, by the Stolz-Cesaro Theorem,

$$\lim_{n\to\infty}(x_n\sqrt{\ln(n)})^2 = \lim_{n\to\infty}\frac{\ln(n)}{\frac{1}{x_n^2}} \stackrel{\text{SC}}{=} \lim_{n\to\infty}\frac{\ln(n+1) - \ln(n)}{\frac{1}{x_{n+1}^2} - \frac{1}{x_n^2}} = \lim_{n\to\infty}\frac{\ln(1+\frac{1}{n})}{\frac{1}{3(n+1)} + \frac{o(1)}{n+1}} = 3$$

Since, $x_n\sqrt{\ln(n)} \geq 0$, we may conclude that $\lim_{n\to\infty} x_n\sqrt{\ln(n)} = \sqrt{3}$. $\square$