Problem 11994
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Let ABC be a triangle with incenter I and circumcircle C. Let M, N , and P be the second points of intersection of Γ with lines AI, BI, and CI, respectively. Let E and F be the points of intersection of NP with AB and AC, respectively. Similarly, let G and H be the points of intersection of MN with AC and BC, respectively, and let J and K be the points of intersection of MP with BC and AB, respectively. Prove

$$|EF| + |GH| + |JK| \leq |KE| + |FG| + |HJ|.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Without loss of generality, we may assume that the circumcircle Γ is the unit circle $|z| = 1$ of the complex plane. We will use the following know properties.

- There exist $u, v, w \in \mathbb{C}$ satisfying

  $$A = u^2,\ B = v^2,\ C = w^2,\ M = -vw, N = -wu, P = -uv.$$

  (M, N and P are the midpoints of the arcs AB, BC, CA, which don't contain C, A, B respectively.)

- If w_1, w_2, w_3 and w_4 lie on the unit circle then the intersection of the segments w_1w_2 and w_3w_4 is given by

  $$f(w_1, w_2, w_3, w_4) = \frac{w_1w_2(w_3 + w_4) - w_3w_4(w_1 + w_2)}{w_1w_2 - w_3w_4}.$$

We have that $a := |BC| = |v^2-w^2|$, $b := |CA| = |w^2-u^2|$, $c := |AB| = |u^2-v^2|$, $m := |PN| = |v-w|$, $n := |MP| = |w - u|$, $p := |MN| = |u - v|$. It can be easily verified that

$$|EF| = \left|-\frac{(v + w)(w + u)(u + v)}{(v - w)}\right| = \frac{abc}{m^2np} \quad \text{and} \quad |FG| = \left|\frac{v(u^2 - w^2)(u + w)}{(v - w)(u - v)}\right| = \frac{b^2}{mnp}$$

where we used the above formula for $E = f(A, B, P, N)$, $F = f(A, C, P, N)$, and $G = f(A, C, N, M)$. Similarly we obtain the other lengths, and then it follows that the desired inequality is equivalent to

$$\frac{1}{m} + \frac{1}{n} + \frac{1}{p} \leq \frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}.$$

Now, since I is the orthocenter of $\triangle MPN$, we have that $\widehat{M} = \frac{\widehat{B}+\widehat{C}}{2}$, and, by the Law of sines,

$$m = 2\sin\left(\frac{\widehat{B} + \widehat{C}}{2}\right), \quad b = 2\sin(\widehat{B}), \quad c = 2\sin(\widehat{C}) \implies m \geq \frac{b + c}{2} \geq \frac{2bc}{b + c}.$$

because the sine function is concave in $[0, \pi]$. In a similar way we have that $n \geq \frac{2ca}{c+a}$, $p \geq \frac{2ab}{a+b}$. Hence

$$\frac{1}{m} + \frac{1}{n} + \frac{1}{p} \leq \frac{b + c}{2bc} + \frac{c + a}{2ca} + \frac{a + b}{2ab} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}$$

where in the last step we used $a^2 + b^2 + c^2 \geq ab + bc + ca$. $\square$