Problem 11993
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Prove that

$$\int_0^1 \frac{\ln(1-x)(\ln(1+x))^2}{x}\,dx = -\frac{\pi^4}{240}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

First solution. By letting $a = \ln(1-x)$ and $b = \ln(1+x)$ in the identity

$$6ab^2 = (a+b)^3 + (a-b)^3 - 2a^3,$$

we get

$$I := \int_0^1 \frac{\ln(1-x)(\ln(1+x))^2}{x}\,dx = \frac{I_1 + I_2 - 2I_3}{6}$$

where

$$I_1 = \int_0^1 \frac{(\ln(1-x^2))^3}{x}\,dx = \frac{1}{2}\int_0^1 \frac{\ln^3(t)}{1-t}\,dt \quad (t = 1-x^2),$$

$$I_2 = \int_0^1 \frac{\left(\ln\left(\frac{1-x}{1+x}\right)\right)^3}{x}\,dx = 2\int_0^1 \frac{\ln^3(t)}{(1-t)(1+t)}\,dt = \int_0^1 \frac{\ln^3(t)}{1-t}\,dt + \int_0^1 \frac{\ln^3(t)}{1+t}\,dt \quad (t = \frac{1-x}{1+x}),$$

$$I_3 = \int_0^1 \frac{(\ln(1-x))^3}{x}\,dx = \int_0^1 \frac{\ln^3(t)}{1-t}\,dt \quad (t = 1-x).$$

Hence, by the uniform convergence,

$$\begin{aligned}
I &= \frac{1}{6}\left(\left(\frac{1}{2} + 1 - 2\right)\int_0^1 \frac{\ln^3(t)}{1-t}\,dt + \int_0^1 \frac{\ln^3(t)}{1+t}\,dt\right) \\
&= \frac{1}{6}\left(-\frac{1}{2}\sum_{n=0}^{\infty}\int_0^1 t^n \ln^3(t)\,dt + \sum_{n=0}^{\infty}(-1)^n \int_0^1 t^n \ln^3(t)\,dt\right) \\
&= \frac{1}{6}\left(\frac{6}{2}\sum_{n=0}^{\infty}\frac{1}{(n+1)^4} - 6\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^4}\right) \\
&= \frac{\zeta(4)}{2} - \frac{7\zeta(4)}{8} = -\frac{3\zeta(4)}{8} = -\frac{3}{8}\cdot\frac{\pi^4}{90} = -\frac{\pi^4}{240}
\end{aligned}$$

where we used the fact that for any non-negative integers m, n,

$$\int_0^1 t^n \ln^m(t)\,dt = \frac{1}{n+1}\left[t^{n+1}\ln^m(t)\right]_0^1 - \frac{m}{n+1}\int_0^1 t^n \ln^{m-1}(t)\,dt = \frac{(-1)^m m!}{(n+1)^{m+1}}.$$

□

Second solution. For $x \in [0,1]$,

$$(\ln(1+x))^2 = 2\sum_{n=1}^{\infty} \frac{H_n(-x)^{n+1}}{n+1}$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Hence, by the uniform convergence,

$$\begin{aligned} I := \int_0^1 \frac{\ln(1-x)(\ln(1+x))^2}{x}\,dx &= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}H_n}{n+1} \int_0^1 x^n \ln(1-x)\,dx \\ &= 2\sum_{n=1}^{\infty} \frac{(-1)^n H_n H_{n+1}}{(n+1)^2} = 2\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}H_n^2}{n^2} - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}H_n}{n^3}\right), \end{aligned}$$

where we used the fact that

$$\begin{aligned} \int_0^1 x^n \ln(1-x)\,dx &= -\frac{1}{n+1}\int_0^1 \ln(1-x)\,d(1-x^{n+1}) \\ &= -\frac{1}{n+1}\left[\ln(1-x)(1-x^{n+1})\right]_0^1 - \frac{1}{n+1}\int_0^1 \frac{1-x^{n+1}}{1-x}\,dx \\ &= -\frac{1}{n+1}\int_0^1 \sum_{k=0}^{n} x^k\,dx = -\frac{H_{n+1}}{n+1}. \end{aligned}$$

Since the values of the two following Euler sums are known,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}H_n}{n^3} &= \frac{41}{16}\zeta(4) - 2\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{7}{4}\log(2)\zeta(3) + \frac{1}{2}\log^2(2)\zeta(2) - \frac{1}{12}\log^4(2), \\ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}H_n^2}{n^2} &= \frac{11}{4}\zeta(4) - 2\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{7}{4}\log(2)\zeta(3) + \frac{1}{2}\log^2(2)\zeta(2) - \frac{1}{12}\log^4(2), \end{aligned}$$

we may conclude that

$$I = 2\left(\frac{41}{16}\zeta(4) - \frac{11}{4}\zeta(4)\right) = -\frac{3}{8}\cdot\frac{\pi^4}{90} = -\frac{\pi^4}{240}.$$

□