Problem 11992
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Prove that, for every positive integer n, there is a positive integer m such that 3^m+5^m-1 is divisible by 7^n.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We will show by induction that for any positive integer n,

$$3^{7^{n-1}} + 5^{7^{n-1}} - 1 \equiv 0 \pmod{7^n}.$$

It trivially holds for $n = 1$. Now we assume that the congruence holds for some $n \geq 1$. Then there is an integer q such that $5^{7^{n-1}} = 1 - x + q7^n$ where $x = 3^{7^{n-1}}$, and

$$\begin{aligned} 3^{7^n} + 5^{7^n} - 1 &= x^7 + (1 - x + q7^n)^7 - 1 \\ &= x^7 + (1-x)^7 - 1 + 7(q7^n)(1-x)^6 + \sum_{k=2}^{7} \binom{7}{k} (q7^n)^k (1-x)^{7-k} \\ &\equiv x^7 + (1-x)^7 - 1 \equiv 7x(x-1)(x^2-x+1)^2 \equiv 0 \pmod{7^{n+1}} \end{aligned}$$

where the last step holds as soon as we show that $x^2 - x + 1 \equiv 0 \pmod{7^n}$.
Note that, since $\varphi(7^n) = 6 \cdot 7^{n-1}$, then, by the Fermat's little theorem,

$$(x-1)(x+1)(x^2+x+1)(x^2-x+1) = x^6 - 1 = 3^{\varphi(7^n)} - 1 \equiv 0 \pmod{7^n}.$$

Moreover $x \equiv 3 \pmod 7$, hence 7 does not divide $(x-1)(x+1)(x^2+x+1) \equiv 6 \pmod 7$, which implies that 7^n divides $x^2 - x + 1$ as claimed. $\square$