Problem 11990
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Proposed by N. Minculete (Romania).

Let a, b, and c be the lengths of the sides of a triangle of area S. Weitzenböck's inequality states that $a^2+b^2+c^2 \geq 4\sqrt{3}S$. Prove the following stronger inequality:

$$a^2+b^2+c^2 \geq \sqrt{3}(4S+(c-a)^2).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We will show the following even stronger inequality

$$a^2+b^2+c^2 \geq 4\sqrt{3}S+2(c-a)^2.$$

By the law of cosines, $b^2 = a^2+c^2-2ac\cos(B)$. Moreover $2S = ac\sin(B)$. Hence

$$\begin{aligned}
a^2+b^2+c^2-4\sqrt{3}S &= 2a^2+2c^2-2ac\cos(B)-2\sqrt{3}ac\sin(B)\\
&= 2(a^2+c^2)-2ac(\cos(B)+\sqrt{3}\sin(B))\\
&= 2(a^2+c^2)-4ac\sin(B+\pi/6)\\
&\geq 2(a^2+c^2)-4ac = 2(c-a)^2 \geq 0.
\end{aligned}$$

□