Problem 11989
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Let x be a number between 0 and 1. Prove

$$\prod_{n=1}^{\infty}(1-x^n) \geq \exp\left(\frac{1}{2}-\frac{1}{2(1-x)^2}\right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $x \in (0,1)$. Then

$$f(x) := -\left(\frac{1}{2}-\frac{1}{2(1-x)^2}\right) = \frac{1}{2}\left(\frac{1}{(1-x)^2}-1\right) = \sum_{k=1}^{\infty}\frac{(k+1)}{2}x^k = \sum_{k=1}^{\infty}\frac{x^k}{k}\sum_{j=1}^{k} j.$$

Moreover

$$g(x) := -\ln\left(\prod_{n=1}^{\infty}(1-x^n)\right) = \sum_{n=1}^{\infty}\ln\left(\frac{1}{1-x^n}\right) = \sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\frac{(x^n)^j}{j} = \sum_{k=1}^{\infty}x^k\sum_{j|k}\frac{1}{j} = \sum_{k=1}^{\infty}\frac{x^k}{k}\sum_{j|k} j.$$

Since $\sum_{j=1}^{k} j \geq \sum_{j|k} j$, it follows that $f(x) \geq g(x)$ and

$$\prod_{n=1}^{\infty}(1-x^n) = \exp(-g(x)) \geq \exp(-f(x)) = \exp\left(\frac{1}{2}-\frac{1}{2(1-x)^2}\right).$$

□

Addendum. Note that for $x \in (0,1)$,

$$g(x) = \sum_{j=1}^{\infty}\frac{1}{j}\sum_{n=1}^{\infty}(x^j)^n = \sum_{j=1}^{\infty}\frac{1}{j}\frac{x^j}{1-x^j} \geq \frac{1}{2}\frac{x^2}{1-x^2} = -\left(\frac{1}{2}-\frac{1}{2(1-x^2)}\right).$$

Hence we have also a *similar* reversed inequality

$$\prod_{n=1}^{\infty}(1-x^n) = \exp(-g(x)) \leq \exp\left(\frac{1}{2}-\frac{1}{2(1-x^2)}\right).$$