Problem 11986
(American Mathematical Monthly, Vol.124, June-July 2017)

Proposed by M. Lukarevski (Macedonia).

Let x, y, and z be positive real numbers. Prove

$$4(xy+yz+zx) \leq (\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x})\sqrt{(x+y)(y+z)(z+x)}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $a=\sqrt{y+z}$, $b=\sqrt{x+z}$, $c=\sqrt{x+y}$, then $a+b>c$, $b+c>a$, $c+a>b$,

$$a^2 \geq a^2-(b-c)^2=(a+b-c)(a+c-b),$$

and similar inequalities hold for b^2 and c^2. Therefore

$$a^2b^2c^2 \geq (a+b-c)^2(a+c-b)^2(b+c-a)^2.$$

Moreover

$$2x=b^2+c^2-a^2, \quad 2y=a^2+c^2-b^2, \quad 2z=a^2+b^2-c^2,$$

which imply

$$\begin{aligned}4(xy+yz+zx) &= (a+b+c)(a+b-c)(a+c-b)(b+c-a)\\ &\leq (a+b+c)abc=(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x})\sqrt{(x+y)(y+z)(z+x)}.\end{aligned}$$

□