Problem 11985
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Proposed by D. Knuth (USA).

For fixed $s, t \in \mathbb{N}$ with $s \leq t$, let $a_n = \sum_{k=s}^{t} \binom{n}{k}$. Prove that this sequence is log-concave, namely that $a_n^2 \geq a_{n-1}a_{n+1}$ for all $n \geq 1$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let

$$F_n(x) := \sum_{k=s}^{t} \binom{n}{k} x^k.$$

Then

$$F_n(x) = \sum_{k=s}^{t} \left(\binom{n-1}{k-1} + \binom{n-1}{k} \right) x^k = \binom{n-1}{s-1} x^s - \binom{n-1}{t} x^{t+1} + (x+1)F_{n-1}(x).$$

Let $P_n(x) := F_n^2(x) - F_{n-1}(x)F_{n+1}(x)$. Then

$$\begin{aligned}
P_n(x) &= F_n(x) \left(\binom{n-1}{s-1} x^s - \binom{n-1}{t} x^{t+1} + (x+1)F_{n-1}(x) \right) \\
&\quad - F_{n-1}(x) \left(\binom{n}{s-1} x^s - \binom{n}{t} x^{t+1} + (x+1)F_n(x) \right) \\
&= \left(\binom{n-1}{s-1} F_n(x) - \binom{n}{s-1} F_{n-1}(x) \right) x^s + \left(\binom{n}{t} F_{n-1}(x) - \binom{n-1}{t} F_n(x) \right) x^{t+1} \\
&= \sum_{k=s}^{t} \left(\binom{n-1}{s-1}\binom{n}{k} - \binom{n}{s-1}\binom{n-1}{k} \right) x^{k+s} + \sum_{k=s}^{t} \left(\binom{n}{t}\binom{n-1}{k} - \binom{n-1}{t}\binom{n}{k} \right) x^{k+t+1}.
\end{aligned}$$

Since $s \leq k \leq t$, it is easy to verify that

$$\binom{n-1}{s-1}\binom{n}{k} \geq \binom{n}{s-1}\binom{n-1}{k} \quad \text{and} \quad \binom{n}{t}\binom{n-1}{k} \geq \binom{n-1}{t}\binom{n}{k}.$$

Hence the polynomial P_n has non-negative coefficients which implies that

$$P_n(1) = F_n^2(1) - F_{n-1}(1)F_{n+1}(1) = a_n^2 - a_{n-1}a_{n+1} \geq 0$$

and the sequence $(a_n)_n$ is log-concave. □