Problem 11984
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Proposed by D. Sitaru (Romania).

Let a, b, and c be the lengths of the sides of a triangle with inradius r. Prove $a^6 + b^6 + c^6 \geq 5184r^6$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $u = b + c - a$, $v = a + c - b$, $w = a + b - c$. Then $u, v, w > 0$ and

$$pr = \sqrt{p(p-a)(p-b)(p-c)} \quad \Longrightarrow \quad 2r = \sqrt{\frac{uvw}{u+v+w}}$$

where $p = (a + b + c)/2$. Moreover $5184 = 2^6 \cdot 3^4$. Hence the inequality is equivalent to

$$\left(\frac{v+w}{2}\right)^6 + \left(\frac{u+w}{2}\right)^6 + \left(\frac{u+v}{2}\right)^6 \geq 3^4 \cdot \frac{u^3v^3w^3}{(u+v+w)^3}$$

or

$$\frac{\left(\frac{v+w}{2}\right)^6 + \left(\frac{u+w}{2}\right)^6 + \left(\frac{u+v}{2}\right)^6}{3} \cdot \left(\frac{u+v+w}{3}\right)^3 \geq u^3v^3w^3$$

which holds because, by the AM-GM inequality,

$$\frac{\left(\frac{v+w}{2}\right)^6 + \left(\frac{u+w}{2}\right)^6 + \left(\frac{u+v}{2}\right)^6}{3} \geq \frac{(vw)^3 + (uw)^3 + (uv)^3}{3} \geq \left((vw)^3 \cdot (uw)^3 \cdot (uv)^3\right)^{1/3} = u^2v^2w^2,$$

and

$$\left(\frac{u+v+w}{3}\right)^3 \geq uvw.$$

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