Problem 11982
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Calculate

$$\lim_{x\to+\infty}\left(\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^n\right)^{1/x}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. It is easy to verify by induction that the known inequalities

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}$$

imply

$$(n-1)!e^{n-1} \le n^n \le n!e^n \quad \text{for any } n \ge 1.$$

Hence

$$\exp(x/e) - 1 = \sum_{n=1}^{\infty}\frac{x^n}{n!e^n} \le \sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^n \le \sum_{n=1}^{\infty}\frac{x^n}{(n-1)!e^{n-1}} = x\exp(x/e),$$

and therefore, for $x > 0$,

$$(\exp(x/e)-1)^{1/x} \le \left(\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^n\right)^{1/x} \le x^{1/x}\exp(1/e).$$

Since

$$\lim_{x\to\infty}(\exp(x/e)-1)^{1/x} = \lim_{x\to\infty}x^{1/x}\exp(1/e) = \exp(1/e),$$

it follows, by the Squeeze Theorem, that the required limit is $\exp(1/e)$. □