Problem 11981
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Proposed by C. Lupu (USA).

Suppose that $f : [0,1] \to \mathbb{R}$ *is a differentiable function with continuous derivative and with*

$$\int_0^1 f(x)dx = \int_0^1 xf(x)dx = 1.$$

Prove that

$$\int_0^1 |f'(x)|^3 dx \geq \left(\frac{128}{3\pi}\right)^2.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. By Hölder inequality,

$$\int_0^1 x(1-x)f'(x)dx \leq \left(\int_0^1 (x(1-x))^{3/2}dx\right)^{2/3} \left(\int_0^1 |f'(x)|^3 dx\right)^{1/3}$$

which implies that

$$\int_0^1 |f'(x)|^3 dx \geq \frac{\left(\int_0^1 x(1-x)f'(x)dx\right)^3}{\left(\int_0^1 (x(1-x))^{3/2}dx\right)^2} = \left(\frac{128}{3\pi}\right)^2$$

because

$$\begin{aligned}\int_0^1 x(1-x)f'(x)dx &= [x(1-x)f(x)]_0^1 - \int_0^1 (x-x^2)'f(x)dx \\ &= 0 - \int_0^1 f(x)dx + 2\int_0^1 xf(x)dx = -1+2 = 1,\end{aligned}$$

and

$$\int_0^1 (x(1-x))^{3/2}dx = B(5/2,5/2) = \frac{\Gamma(5/2)^2}{\Gamma(5)} = \frac{3\pi}{128}.$$

□