Problem 11979
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Proposed by Z. Franco (USA).

Let O and I denote the circumcenter and incenter of a triangle ABC. Are there infinitely many nonsimilar scalene triangles ABC for which the lengths AB, BC, CA, and OI are all integers?

Solution proposed by Robin Chapman, Department of Mathematics, University of Exeter, UK, and Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", Italy.

Solution. A formula of Euler states that

$$|OI|^2 = R(R-2r)$$

where R and r are the circumradius and the inradius of the triangle ABC. Now

$$R = \frac{abc}{4\Delta} \quad \text{and} \quad r = \frac{\Delta}{s}$$

where Δ is the area of ABC and s its semiperimeter. Thus

$$|OI|^2 = \frac{a^2b^2c^2}{16\Delta^2} - \frac{abc}{2s} = \frac{abc}{2s}\left(\frac{abc}{8(s-a)(s-b)(s-c)} - 1\right)$$

using Heron's formula.
Let x, y be positive integers with $y > 1$, such that $x^2 - 3y^2 = 1$. Then by letting

$$a := (y+1)(3y-1), \quad b := 3y^2+1, \quad c := 4y$$

we have that $a > b > c$, $b + c > a$ and

$$|IO| = (y-1)\sqrt{3y^2+1} = (y-1)x \in \mathbb{N}^+.$$

Notice that $b^2 = a^2 + c^2 - ca$ implies that the angle B is 60 degrees.
The Pell's equation $x^2 - 3y^2 = 1$ has infinitely many solutions (x_n, y_n): $x_1 := 7$, $y_1 := 4$, and for $n \geq 1$,

$$x_{n+1} := 2x_n + 3y_n, \qquad y_{n+1} := x_n + 2y_n.$$

Therefore the infinitely many integer triangles $(a_n, b_n, c_n) = ((y_n+1)(3y_n-1), 3y_n^2+1, 4y_n)$ are scalene, where the distance $|OI| = (y_n - 1)x_n$ is a positive integer. All these triangles are nonsimilar because

$$\frac{b_n}{c_n} = \frac{3y_n^2+1}{4y_n} = \frac{1}{4}\left(3y_n + \frac{1}{y_n}\right)$$

is a strictly increasing sequence (due to the fact that $(y_n)_n$ is strictly increasing too). $\square$