Problem 11978
(American Mathematical Monthly, Vol.124, May 2017)

Proposed by H. Ohtsuka (Japan).

Let F_n be the n-th Fibonacci number. Find

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\cosh(F_n)\cosh(F_{n+3})}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $u_n = 2\cosh(F_n)$ then

$$\begin{aligned} u_{n+1}u_{n+2} &= \left(e^{F_{n+1}} + e^{-F_{n+1}}\right)\left(e^{F_{n+2}} + e^{-F_{n+2}}\right) \\ &= e^{F_{n+1}+F_{n+2}} + e^{F_{n+2}-F_{n+1}} + e^{-F_{n+2}+F_{n+1}} + e^{-F_{n+1}-F_{n+2}} \\ &= e^{F_{n+3}} + e^{F_n} + e^{-F_n} + e^{-F_{n+3}} \\ &= u_n + u_{n+3}. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{n=0}^{N} \frac{(-1)^n}{\cosh(F_n)\cosh(F_{n+3})} &= 4\sum_{n=0}^{N} \frac{(-1)^n}{u_n u_{n+3}} = 4\sum_{n=0}^{N} \frac{(-1)^n(u_n + u_{n+3})}{u_n u_{n+1} u_{n+2} u_{n+3}} \\ &= 4\sum_{n=0}^{N} \left(\frac{(-1)^n}{u_{n+1}u_{n+2}u_{n+3}} - \frac{(-1)^{n-1}}{u_n u_{n+1} u_{n+2}}\right) \\ &= 4\left(\frac{(-1)^N}{u_{N+1}u_{N+2}u_{N+3}} - \frac{-1}{u_0 u_1 u_2}\right) \stackrel{N\to\infty}{\to} \frac{4}{u_0 u_1 u_2} = \frac{1}{2\cosh^2(1)}. \end{aligned}$$

□