Problem 11976
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Given a positive real number s, consider the sequence $\{u_n\}_{n\geq 1}$ defined by $u_1 = 1$, $u_2 = s$, and $u_{n+2} = u_n u_{n+1}/n$ for $n \geq 1$.
(a) Show that there is a constant C such that $\lim_{n\to\infty} u_n = \infty$ when $s > C$ and $\lim_{n\to\infty} u_n = 0$ when $s < C$.
(b) Calculate $\lim_{n\to\infty} u_n$ when $s = C$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $F_n = (\alpha^n - \beta^n)/\sqrt{5}$ be the nth-Fibonacci number where $\alpha = (1+\sqrt{5})/2$ and $\beta = -1/\alpha$. Then, by using the relation $F_{n+2} = F_n + F_{n+1}$, it is to verify by induction that

$$u_{n+1} = \frac{s^{F_n}}{\prod_{k=1}^{n} k^{F_{n-k}}}.$$

Hence

$$\begin{aligned}\sqrt{5}\ln(u_{n+1}) =& \sqrt{5}\left(F_n \ln(s) - \sum_{k=1}^{n} F_{n-k}\ln(k)\right)\\ &= (\alpha^n - \beta^n)\ln(s) + \sum_{k=1}^{n}\alpha^{n-k}\ln(k) - \sum_{k=1}^{n}\beta^{n-k}\ln(k)\\ &= \alpha^n(\ln(s) - L) + \gamma_n - \beta^n \ln(s)\end{aligned}$$

where

$$L := \sum_{k=1}^{\infty}\alpha^{-k}\ln(k) \approx 1.163451 \quad \text{and} \quad \gamma_n := \sum_{k=n+1}^{\infty}\alpha^{n-k}\ln(k) + \sum_{k=1}^{n}\beta^{n-k}\ln(k)$$

(the series is convergent because the radius of convergence at 0 of the power series $\sum_{k=1}^{\infty} x^k \ln(k)$ is 1 and $\alpha^{-1} \in (0,1)$).

(a) As n goes to infinity, we have that $\beta^n \ln(s) \to 0$, and

$$0 \leq \frac{|\gamma_n|}{\alpha^n} \leq \sum_{k=n+1}^{\infty}\alpha^{-k}\ln(k) + \frac{n\ln(n)}{\alpha^n} \to 0.$$

Hence, by letting $C = e^L \approx 3.20096065$, we have

$$\sqrt{5}\ln(u_{n+1}) = \alpha^n(\ln(s/C) + o(1)) + o(1).$$

Therefore if $s > C$ then $\ln(s/C) > 0$ and $\lim_{n\to\infty} u_n = \infty$. On the other hand, $\lim_{n\to\infty} u_n = 0$ when $s < C$.

(b) If $s = C$ then $\sqrt{5}\ln(u_{n+1}) = \gamma_n + o(1)$ which implies that $\lim_{n\to\infty} u_n = \infty$ because

$$\begin{aligned}\gamma_n &\geq \sum_{k=n+1}^{2n-1}\alpha^{n-k}\ln(k) + \sum_{k=1}^{n}\beta^{n-k}\ln(k) = \sum_{k=1}^{n-1}\alpha^{-k}\ln(n+k) + \sum_{k=0}^{n-1}(-\alpha)^{-k}\ln(n-k)\\ &= \ln(n) + \sum_{k=1}^{n-1}\alpha^{-k}\left(\ln(n+k) + (-1)^k\ln(n-k)\right) \geq \ln(n).\end{aligned}$$

□