Problem 11975
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Proposed by I. Mező (China).

Let x be a real number in $[0,1)$. Prove that

$$\frac{(1-\gamma)^x}{1-x} \leq \int_0^1 \Gamma^x(t)\,dt \leq \frac{1}{1-x}$$

where Γ is the gamma function and γ is the Euler-Mascheroni constant.

Solution proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France, and Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", Italy.

Solution. By the log-convexity of the gamma function on the positive real axis, it follows that for $t \in [0,1]$,

$$0 = \log(\Gamma(2)) = \log(\Gamma(t(t+1) + (1-t)(t+2)) \leq t\log(\Gamma(t+1)) + (1-t)\log(\Gamma(t+2))$$

that is, since $\Gamma(t+2) = (t+1)\Gamma(t+1)$,

$$1 \leq \Gamma^t(t+1)\Gamma^{1-t}(t+2) = (t+1)^{1-t}\Gamma(t+1)$$

and

$$m := \min_{t\in[0,1]} \frac{1}{(t+1)^{1-t}} \leq \Gamma(t+1).$$

On the other hand, log-convexity implies convexity and therefore for $t \in [0,1]$,

$$\Gamma(t+1) \leq \max\{\Gamma(0+1), \Gamma(1+1)\} = 1.$$

Hence, putting all together we get

$$m \leq \Gamma(t+1) = t\Gamma(t) \leq 1$$

and, for $x \in [0,1)$, we have

$$\frac{m^x}{t^x} \leq \Gamma^x(t) \leq \frac{1}{t^x}.$$

Finally, by integration with respect to t over $[0,1]$, we obtain

$$\frac{m^x}{1-x} = m^x \int_0^1 \frac{1}{t^x}\,dt \leq \int_0^1 \Gamma^x(t)\,dt \leq \int_0^1 \frac{1}{t^x}\,dt = \frac{1}{1-x}$$

which is a more general result because $m \approx 0.815154 > 1/2 > (1-\gamma)$. $\square$