Problem 11973
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Proposed by D. Orr (USA).

Prove

$$G = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)4^n}\left(1-\frac{2}{4^n}\right),$$

where $G = \sum_{n=0}^{\infty}(-1)^n/(2n+1)^2$ *(Catalan's constant),* $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ *for* $s > 1$*, and* $\zeta(0) = -1/2$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. It is known that

$$\begin{aligned}(\pi x)\cot(\pi x) &= 1 + 2x^2\sum_{k=1}^{\infty}\frac{1}{x^2-k^2} = 1 - 2\sum_{k=1}^{\infty}\frac{(x/k)^2}{1-(x/k)^2}\\ &= 1 - 2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{x^{2n}}{k^{2n}} = 1 - 2\sum_{n=1}^{\infty}\zeta(2n)x^{2n} = -2\sum_{n=0}^{\infty}\zeta(2n)x^{2n}.\end{aligned}$$

Hence

$$F(x) := \int_0^{\pi x} t\cot(t)\,dt = \pi\int_0^x (\pi s)\cot(\pi s)\,ds = -2\pi\sum_{n=0}^{\infty}\frac{\zeta(2n)x^{2n+1}}{2n+1}.$$

Moreover, for $x \in [0,1]$,

$$F(x) = -\pi x C_1(2\pi x) + \frac{1}{2}S_2(2\pi x)$$

where

$$\begin{aligned}C_1(\theta) &= \sum_{k=1}^{\infty}\frac{\cos(k\theta)}{k} = \sum_{k=1}^{\infty}\frac{e^{ik\theta}+e^{-ik\theta}}{2k} = -\frac{1}{2}\ln(1-e^{i\theta}) - \frac{1}{2}\ln(1-e^{-i\theta})\\ &= -\frac{1}{2}\ln(2-2\cos(\theta)) = -\ln(|2\sin(\theta/2)|),\end{aligned}$$

and

$$S_2(\theta) = \sum_{k=1}^{\infty}\frac{\sin(k\theta)}{k^2}.$$

In fact the above identity holds because l.h.s and r.h.s. agree at $x = 0$, and for $x \in (0,1)$, the derivative of the l.h.s., $F'(x) = \pi^2 x\cot(\pi x)$, coincides with the derivative of the r.h.s.,

$$\begin{aligned}\frac{d}{dx}\left(-\pi x C_1(2\pi x) + \frac{S_2(2\pi x)}{2}\right) &= -\pi C_1(2\pi x) - 2\pi^2 x C_1'(2\pi x) + \pi C_2'(2\pi x)\\ &= -\pi C_1(2\pi x) - 2\pi^2 x(-\cot(\pi x)/2) + \pi C_1(2\pi x) = \pi^2 x\cot(\pi x).\end{aligned}$$

Hence

$$\begin{aligned}\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)4^n}\left(1-\frac{2}{4^n}\right) &= \pi\sum_{n=0}^{\infty}\frac{\zeta(2n)(1/2)^{2n+1}}{(2n+1)} - 4\pi\sum_{n=0}^{\infty}\frac{\zeta(2n)(1/4)^{2n+1}}{(2n+1)}\\ &= -\frac{F(1/2)}{2} + 2F(1/4)\\ &= \frac{\pi C_1(\pi)}{4} - \frac{S_2(\pi)}{4} - \frac{\pi C_1(\pi/2)}{2} + S_2(\pi/2)\\ &= \frac{\pi(-\ln(2))}{4} - 0 - \frac{\pi(-\ln(\sqrt{2}))}{2} + G = G.\end{aligned}$$

□