Problem 11972
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Proposed by Yun Zhang (China).

Let r be the radius of the sphere inscribed in a tetrahedron whose exscribed spheres have radii r_1, r_2, r_3, and r_4. Prove

$$r(\sqrt[3]{r_1}+\sqrt[3]{r_2}+\sqrt[3]{r_3}+\sqrt[3]{r_4}) \leq 2\sqrt[3]{r_1r_2r_3r_4}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let us consider an n-simplex in $\mathbb{R}^n$ for $n \geq 2$. We show that

$$r\sum_{k=1}^{n+1}\sqrt[n]{r_k} \leq (n-1)\sqrt[n]{\prod_{k=1}^{n+1} r_k},$$

where r is the radius of the inscribed sphere and $r_1, r_2, \ldots, r_{n+1}$ are the radii of the $n+1$ exscribed spheres. The required inequality can be obtained by setting $n = 3$.

We have that

$$\frac{r\sum_{k=1}^{n+1}\sqrt[n]{r_k}}{\sqrt[n]{\prod_{k=1}^{n+1} r_k}} = \sum_{k=1}^{n+1}\sqrt[n]{\prod_{j\neq k}\frac{r}{r_j}} \leq \sum_{k=1}^{n+1}\frac{1}{n}\sum_{j\neq k}\frac{r}{r_j} = r\sum_{k=1}^{n+1}\frac{1}{r_k} = n-1$$

where in the second step we applied the AM-GM inequality and in the last step we used the known relation

$$r\sum_{k=1}^{n+1}\frac{1}{r_k} = n-1.$$

Remark For the sake of completeness, we give here a proof of the above known relation (see for example A.A. Toda, *Radii of the inscribed and escribed spheres of a simplex*, Int. J. Geom. 3, No. 2, 5-13 (2014)).
Let $S = \{P_1, P_2, \ldots, P_{n+1}\}$ be the vertices of the n-simplex, and let $S_k = S \setminus \{P_k\}$ be the vertices of the face opposite to the vertex P_k. Then, for $1 \leq k \leq n+1$,

$$\operatorname{conv}(\{I_k\}\cup S_k) \cup \bigcup_{j=1}^{n+1}\operatorname{conv}(\{I\}\cup S_j) = \operatorname{conv}(\{I_k\}\cup S) = \bigcup_{j\neq k}\operatorname{conv}(\{I_k\}\cup S_j)$$

which implies, by computing the n-dimensional volumes (the sets are pairwise disjoint),

$$\frac{r_k|S_k|}{n} + \sum_{j=1}^{n+1}\frac{r|S_j|}{n} = \sum_{j\neq k}\frac{r_k|S_j|}{n} \implies \frac{r}{r_k} = 1 - \frac{2|S_k|}{\sum_{j=1}^{n+1}|S_j|}.$$

Hence

$$r\sum_{k=1}^{n+1}\frac{1}{r_k} = \sum_{k=1}^{n+1}\left(1 - \frac{2|S_k|}{\sum_{j=1}^{n+1}|S_j|}\right) = (n+1) - 2 = n-1.$$

□