Problem 11971
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Proposed by S. P. Andriopoulos (Greece).

For $n \geq 2$, let $a_1, \dots, a_n$ be positive real numbers. Prove

$$\left(\prod_{i=1}^{n}(1+a_i)\right)^{n-1} \geq \left(\prod_{1\leq i<j\leq n}\left(1+\frac{2a_ia_j}{a_i+a_j}\right)\right)^2.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We first note that if $x, y > 0$ then

$$(1+x)(1+y) - \left(1+\frac{2xy}{x+y}\right)^2 = \frac{(x+xy+y)(x-y)^2}{(x+y)^2} \geq 0.$$

Hence

$$\left(\prod_{i=1}^{n}(1+a_i)\right)^{n-1} = \prod_{1\leq i<j\leq n}(1+a_i)(1+a_j) \geq \prod_{1\leq i<j\leq n}\left(1+\frac{2a_ia_j}{a_i+a_j}\right)^2 = \left(\prod_{1\leq i<j\leq n}\left(1+\frac{2a_ia_j}{a_i+a_j}\right)\right)^2.$$

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