Problem 11969
(American Mathematical Monthly, Vol.124, March 2017)

Proposed by A. Dzhumadil'daev (Kazakhstan).

Let $x_1, \dots, x_n$ be indeterminates, and let A be the n-by-n matrix with (i,j) entry $1/\cos(x_i - x_j)$. Prove

$$\det(A) = (-1)^{\binom{n}{2}} \prod_{1\leq i<j\leq n} \tan^2(x_i - x_j).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We will use the determinant identity

$$\det\left(\left[\frac{1}{a_i+b_j}\right]_{n\times n}\right) = \frac{\prod_{1\leq i<j\leq n}(a_i-a_j)(b_i-b_j)}{\prod_{1\leq i,j\leq n}(a_i+b_j)}$$

which appears in T. Muir's book *A Treatise on the Theory of Determinants*, par. 353, p. 348 (see also Monthly problem 10387). The identity can be proved by induction by noting that if we subtract to the i-th row the n-th row multiplied by $(a_n+b_n)/(a_i+b_n)$ for $i=1,\dots,n$, we get

$$\det\left(\left[\frac{1}{a_i+b_j}\right]_{n\times n}\right) = \frac{1}{a_n+b_n}\det\left(\left[\frac{(a_i-a_n)(b_j-b_n)}{(a_i+b_j)(a_i+b_n)(a_n+b_j)}\right]_{n-1\times n-1}\right)$$
$$= \frac{\prod_{1\leq i\leq n-1}(a_i-a_n)(b_i-b_n)}{(a_n+b_n)\prod_{1\leq i\leq n-1}(a_i+b_n)(a_n+b_i)}\det\left(\left[\frac{1}{a_i+b_j}\right]_{n-1\times n-1}\right).$$

Let $a_i = b_i = e^{2ix_i}$ then

$$\cos(x_i-x_j) = \frac{e^{i(x_i-x_j)}+e^{-i(x_i-x_j)}}{2} = \frac{a_i+a_j}{2e^{ix_i}e^{ix_j}}$$

and

$$\tan(x_i-x_j) = \frac{\sin(x_i-x_j)}{\cos(x_i-x_j)} = \frac{e^{i(x_i-x_j)}-e^{-i(x_i-x_j)}}{i(e^{i(x_i-x_j)}+e^{-i(x_i-x_j)})} = \frac{a_i-a_j}{i(a_i+a_j)}.$$

Hence, by Muir's identity,

$$\det(A) = \det\left(\left[\frac{2e^{ix_i}e^{ix_j}}{a_i+a_j}\right]\right) = 2^n\prod_{i=1}^n e^{ix_i}\prod_{j=1}^n e^{ix_j}\cdot\det\left(\left[\frac{1}{a_i+a_j}\right]\right)$$
$$= \prod_{i=1}^n(2a_i)\cdot\frac{\prod_{1\leq i<j\leq n}(a_i-a_j)^2}{\prod_{1\leq i,j\leq n}(a_i+a_j)} = \frac{\prod_{1\leq i<j\leq n}(a_i-a_j)^2}{\prod_{1\leq i<j\leq n}(a_i+a_j)^2}$$
$$= \prod_{1\leq i<j\leq n}\left(\frac{a_i-a_j}{a_i+a_j}\right)^2 = \prod_{1\leq i<j\leq n}(i\tan(x_i-x_j))^2$$
$$= (-1)^{\binom{n}{2}}\prod_{1\leq i<j\leq n}\tan^2(x_i-x_j).$$

□