Problem 11968
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For $n \geq 1$, prove that $F_{5n}/(5F_n)$ is an integer congruent to 1 modulo 10 where F_n is the nth Fibonacci number.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. It suffices to show that

$$\frac{F_{5n}}{5F_n} = 5F_n^2(F_n^2 + (-1)^n) + 1.$$

In fact, the right-hand side is an integer and

$$5F_n^2(F_n^2 + (-1)^n) + 1 \equiv 5F_n^2(F_n^2 + 1) + 1 \equiv 1 \pmod{10}$$

because the product of two consecutive integers $F_n^2(F_n^2 + 1)$ is even.
Let $\varphi_\pm = (1 \pm \sqrt{5})/2$, $x = \varphi_+^n$ and $y = \varphi_-^n$ then $xy = (-1)^n$ and $F_n = (x - y)/\sqrt{5}$.
Moreover

$$5F_n^2 = (x-y)^2 = x^2 - 2xy + y^2 = x^2 + y^2 - 2(-1)^n \Rightarrow x^2 + y^2 = 5F_n^2 + 2(-1)^n,$$

and

$$\begin{aligned} 25F_n^4 &= (x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 \\ &= x^4 + y^4 - 4(-1)^n(x^2 + y^2) + 6 \Rightarrow x^4 + y^4 = 25F_n^4 + 4(-1)^n(x^2 + y^2) - 6. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{F_{5n}}{F_n} &= \frac{x^5 - y^5}{x - y} = x^4 + x^3y + x^2y^2 + xy^3 + y^4 = (x^4 + y^4) + (-1)^n(x^2 + y^2) + 1 \\ &= 25F_n^4 + 5(-1)^n(x^2 + y^2) - 5 = 25F_n^4 + 5(-1)^n(5F_n^2 + 2(-1)^n) - 5 \\ &= 25F_n^4 + 25(-1)^nF_n^2 + 5 = 5\left[5F_n^2(F_n^2 + (-1)^n) + 1\right] \end{aligned}$$

and the proof of the required identity is complete. □