Problem 11967
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Let F_n be the n-th Fermat number $2^{2^n}+1$. Find

$$\lim_{n\to\infty}\sqrt{6F_1+\sqrt{6F_2+\sqrt{6F_3+\sqrt{\cdots+\sqrt{6F_n}}}}}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. For $x\geq 0$, we consider the sequence of functions defined by the recurrence

$$f_0(x)=0,\quad f_{n+1}(x)=\sqrt{6(1+x)+f_n(x^2)}\quad\text{for } n\geq 0.$$

We show by induction that for $n\geq 2$, and for $x\in[0,1/2]$,

$$a_n+\frac{3x}{a_n}\leq f_n(x)\leq 3+x$$

where $a_n=f_n(0)$. Note that the recurrence $a_{n+1}=\sqrt{6+a_n}$ implies that the sequence $(a_n)_{n\geq 2}$ is positive, increasing and it tends to 3. Therefore, by the double inequality, for any $x\in[0,1/2]$,

$$\lim_{n\to\infty} f_n(x)=3+x.$$

The base case $n=2$ can be easily verified. Inductive step: let $n\geq 2$, then

$$f_{n+1}(x)=\sqrt{6(1+x)+f_n(x^2)}\leq\sqrt{6(1+x)+(3+x^2)}=3+x.$$

Moreover

$$f_{n+1}(x)=\sqrt{6(1+x)+f_n(x^2)}\geq\sqrt{6(1+x)+\left(a_n+\frac{3x^2}{a_n}\right)}\geq a_{n+1}+\frac{3x}{a_{n+1}}$$

where the last inequality is equivalent to

$$6+6x+a_n+\frac{3x^2}{a_n}\geq a_{n+1}^2+6x+\frac{9x^2}{a_{n+1}^2}=6+a_n+6x+\frac{9x^2}{6+a_n}$$

which holds because $a_n\leq 3$, and the proof of the double inequality is complete.
Finally, we are able to evaluate the required limit. Since

$$\sqrt{6F_1+\sqrt{6F_2+\sqrt{6F_3+\sqrt{\cdots+\sqrt{6F_n}}}}}=2f_n(1/4),$$

it follows that, as n goes to infinity, the limit of the left-hand side is $2(3+1/4)=13/2$. $\square$