Problem 11966
(American Mathematical Monthly, Vol.124, March 2017)

Proposed by C. I. Vălean (Romania).

Prove that

$$\int_0^1 \frac{x\ln(1+x)}{1+x^2}\,dx = \frac{\pi^2}{96} + \frac{(\ln(2))^2}{8}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let

$$F(a) = \int_0^1 \frac{x\ln(1+ax)}{1+x^2}\,dx,$$

then the derivative is

$$\begin{aligned}
F'(a) &= \int_0^1 \frac{x^2}{(1+x^2)(1+ax)}\,dx = \frac{1}{a^2+1}\int_0^1 \left(\frac{ax-1}{1+x^2} + \frac{1}{1+ax}\right) dx \\
&= \frac{1}{a^2+1}\left[\frac{a}{2}\ln(1+x^2) - \arctan(x) + \frac{\ln(1+ax)}{a}\right]_0^1 \\
&= \frac{1}{a^2+1}\left(\frac{a}{2}\ln(2) - \frac{\pi}{4} + \frac{\ln(1+a)}{a}\right) \\
&= \frac{\frac{1}{2}\ln(2)a}{a^2+1} - \frac{\frac{\pi}{4}}{a^2+1} - \frac{a\ln(1+a)}{a^2+1} + \frac{\ln(1+a)}{a}.
\end{aligned}$$

Let

$$I := \int_0^1 \frac{x\ln(1+x)}{1+x^2}\,dx,$$

then it follows that

$$\begin{aligned}
I = F(1) &= F(0) + \int_0^1 F'(a)\,da \\
&= 0 + \frac{1}{2}\ln(2)\int_0^1 \frac{a}{a^2+1}\,da - \frac{\pi}{4}\int_0^1 \frac{1}{a^2+1}\,da - I + \int_0^1 \frac{\ln(1+a)}{a}\,da \\
&= \frac{1}{2}\ln(2)\left[\frac{\ln(a^2+1)}{2}\right]_0^1 - \frac{\pi}{4}\left[\arctan(a)\right]_0^1 - I + \frac{\pi^2}{12} \\
&= -I + \frac{\pi^2}{48} + \frac{(\ln(2))^2}{4},
\end{aligned}$$

where we used the known fact

$$\int_0^1 \frac{\ln(1+a)}{a}\,da = \sum_{k=1}^{\infty}(-1)^{k+1}\int_0^1 \frac{a^{k-1}}{k}\,da = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}.$$

Therefore, we may conclude

$$I = \frac{\pi^2}{96} + \frac{(\ln(2))^2}{8}.$$

□