Problem 11963
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Proposed by G. Alexe and G.F. Serban (Romania).

Let $a_1, \ldots, a_n$ be positive real numbers with $\prod_{i=1}^n a_i = 1$. Show that

$$\sum_{i=1}^n \frac{(a_i + a_{i+1})^4}{a_i^2 - a_i a_{i+1} + a_{i+1}^2} \geq 12n,$$

where $a_{n+1} = a_1$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We have that

$$\frac{(x+y)^4}{x^2 - xy + y^2} - 12xy = \frac{(x+y)^4 - 12xy(x^2 - xy + y^2)}{x^2 - xy + y^2} = \frac{(x^2 - 4xy + y^2)^2}{x^2 - xy + y^2} \geq 0.$$

Hence

$$\sum_{i=1}^n \frac{(a_i + a_{i+1})^4}{a_i^2 - a_i a_{i+1} + a_{i+1}^2} \geq 12 \sum_{i=1}^n a_i a_{i+1} \geq 12n \left(\prod_{i=1}^n a_i a_{i+1} \right)^{1/n} = 12n \left(\prod_{i=1}^n a_i \right)^{2/n} = 12n.$$

where in the second step we used the AM-GM inequality.

Note that 12 is the best constant because equality holds when $a_1 = \sqrt{3/2} + \sqrt{1/2}$, $a_2 = \sqrt{3/2} - \sqrt{1/2}$.
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