Problem 11961
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Proposed by M. Berindeanu (Romania).

Evaluate

$$\int_0^{\pi/2} \frac{\sin(x)}{1+\sqrt{\sin(2x)}}\,dx.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. By letting $t = x - \pi/4$, and $\sqrt{2}\sin(t) = \sin(\theta)$, we have that

$$\begin{aligned}
I &:= \int_0^{\pi/2} \frac{\sin(x)}{1+\sqrt{\sin(2x)}}\,dx = \int_{-\pi/4}^{\pi/4} \frac{\sin(t+\pi/4)}{1+\sqrt{\sin(2t+\pi/2)}}\,dt \\
&= \frac{1}{\sqrt{2}}\int_{-\pi/4}^{\pi/4} \frac{\sin(t)+\cos(t)}{1+\sqrt{\cos(2t)}}\,dt = 0 + \frac{2}{\sqrt{2}}\int_0^{\pi/4} \frac{\cos(t)}{1+\sqrt{1-2\sin^2(t)}}\,dt \\
&= \int_0^{\pi/4} \frac{\sqrt{2}\cos(t)dt}{1+\sqrt{1-(\sqrt{2}\sin(t))^2}} = \int_0^{\pi/2} \frac{\cos(\theta)d\theta}{1+\cos(\theta)} \\
&= \int_0^{\pi/2} \left(1 - \frac{1}{1+\cos(\theta)}\right)d\theta = [\theta - \tan(\theta/2)]_0^{\pi/2} = \frac{\pi}{2} - 1.
\end{aligned}$$

□