Problem 11959
(American Mathematical Monthly, Vol.124, February 2017)

Proposed by D. Knuth (USA).

Prove that, for any n-by-n matrix A with (i,j)-entry $a_{i,j}$ and any $t_1,\dots,t_n$, the permanent of A is

$$\frac{1}{2^n}\sum\prod_{i=1}^{n}\sigma_i\left(t_i+\sum_{j=1}^{n}\sigma_j a_{i,j}\right),$$

where the outer sum is over all 2^n choices of $(\sigma_1,\dots,\sigma_n)\in\{1,-1\}^n$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let

$$F_A(t_1,\dots,t_n):=\frac{1}{2^n}\sum_{\sigma\in\{1,-1\}^n}\prod_{i=1}^{n}\sigma_i\left(t_i+\sum_{j=1}^{n}\sigma_j a_{i,j}\right).$$

We will show that $F_A(t_1,\dots,t_n)=\mathrm{perm}(A)$ by induction on n.
If $n=1$ then the formula holds:

$$F_A(t_1)=\frac{1}{2}\left(1\cdot(t_1+1\cdot a_{1,1})+(-1)\cdot(t_1+(-1)\cdot a_{1,1})\right)=a_{1,1}=\mathrm{perm}(A).$$

Assume that $n>1$, then we first prove that the multivariable polynomial F_A is constant by showing that the partial derivatives are identically zero. By symmetry, it suffices to check the partial derivative with respect to t_n,

$$\begin{aligned}\frac{\partial F_A}{\partial t_n}(t_1,\dots,t_n)&=\frac{1}{2^n}\sum_{\sigma\in\{1,-1\}^n}\sigma_n\prod_{i=1}^{n-1}\sigma_i\left(t_i+\sum_{j=1}^{n}\sigma_j a_{i,j}\right)\\&=\frac{1}{2^n}\sum_{\sigma\in\{1,-1\}^{n-1}}1\cdot\prod_{i=1}^{n-1}\sigma_i\left(t_i+1\cdot a_{i,n}+\sum_{j=1}^{n-1}\sigma_j a_{i,j}\right)\\&\quad+\frac{1}{2^n}\sum_{\sigma\in\{1,-1\}^{n-1}}(-1)\cdot\prod_{i=1}^{n-1}\sigma_i\left(t_i+(-1)\cdot a_{i,n}+\sum_{j=1}^{n-1}\sigma_j a_{i,j}\right)\\&=\frac{F_{A'}(t_1+a_{1,n},\dots,t_{n-1}+a_{n-1,n})-F_{A'}(t_1-a_{1,n},\dots,t_{n-1}-a_{n-1,n})}{2}\\&=\frac{\mathrm{perm}(A')-\mathrm{perm}(A')}{2}=0.\end{aligned}$$

where A' is the $(n-1)$-by-$(n-1)$ matrix obtained by removing both the n-th row and the n-th column of A.
Hence, it remains to show that $F_A(0,\dots,0)=\mathrm{perm}(A)$. Let $\mathcal{F}_n$ be the set of functions of $\{1,\dots,n\}$ in itself, then

$$\begin{aligned}F_A(0,\dots,0)&=\frac{1}{2^n}\sum_{\sigma\in\{1,-1\}^n}\prod_{i=1}^{n}\left(\sum_{j=1}^{n}\sigma_i\sigma_j a_{i,j}\right)=\frac{1}{2^n}\sum_{\sigma\in\{1,-1\}^n}\sum_{f\in\mathcal{F}_n}\prod_{i=1}^{n}\left(\sigma_i\sigma_{f(i)}a_{i,f(i)}\right)\\&=\sum_{f\in\mathcal{F}_n}\frac{1}{2^n}\sum_{\sigma\in\{1,-1\}^n}\prod_{i=1}^{n}\left(\sigma_i\sigma_{f(i)}a_{i,f(i)}\right)=\mathrm{perm}(A)\end{aligned}$$

because

$$\frac{1}{2^n}\sum_{\sigma\in\{1,-1\}^n}\prod_{i=1}^{n}\left(\sigma_i\sigma_{f(i)}a_{i,f(i)}\right)=\begin{cases}\prod_{i=1}^{n}a_{i,f(i)} & \text{if } f \text{ is a permutation,}\\ 0 & \text{otherwise.}\end{cases}$$

□