Problem 11957
(American Mathematical Monthly, Vol.124, February 2017)

Proposed by E. Pité (France).

Let k and n be two integers with $n \geq k \geq 2$. Let $S(n,k)$ be the Stirling number of the second kind, i.e., the number of ways to partition a set of n objects into k nonempty subsets. Show that

$$n^k S(n,k) \geq k^n \binom{n}{k}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We will show that the inequality holds for $n \geq k \geq 1$ by double induction.
Basic step: since $S(n,1) = n$ then the inequality holds trivially for $n \geq k = 1$.
Assume now that $n \geq k \geq 1$. Note that the following function is convex: for $x > 0$,

$$f_k(x) := \frac{1}{\left(1+\frac{x}{k}\right)^k} \qquad \text{implies} \qquad f_k''(x) = \frac{1+\frac{1}{k}}{\left(1+\frac{x}{k}\right)^{k+2}} > 0.$$

By a known identity, by the induction hypothesis, and by the convexity of f_k, we have

$$\begin{aligned} S(n+1,k+1) &= \sum_{j=k}^{n} \binom{n}{j} S(j,k) \geq \sum_{j=k}^{n} \binom{n}{j}\binom{j}{k}\frac{k^j}{j^k} = \binom{n}{k}\sum_{J=0}^{N}\binom{N}{J} k^J f_k(J) \\ &\geq \binom{n}{k}(k+1)^N f_k\left(\frac{1}{(k+1)^N}\sum_{J=0}^{N}\binom{N}{J}k^J J\right) = \binom{n}{k}(k+1)^N f_k\left(\frac{kN}{k+1}\right) \\ &= \binom{n}{k}\frac{(k+1)^N}{\left(1+\frac{N}{k+1}\right)^k} = \binom{n}{k}\frac{(k+1)^{N+k}}{(N+k+1)^k} = \binom{n+1}{k+1}\frac{(k+1)^{n+1}}{(n+1)^{k+1}} \end{aligned}$$

where $N = n-k$, $J = j-k$, and it can be easily verified that

$$\frac{1}{(k+1)^N}\sum_{J=0}^{N}\binom{N}{J}k^J = 1, \quad \text{and} \quad \frac{1}{(k+1)^N}\sum_{J=0}^{N}\binom{N}{J}k^J J = \frac{kN}{k+1}.$$

Therefore the proof of the induction step is complete. □