**Problem 11956**
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Proposed by H. Ohtsuka (Japan).

*Show that*

$$\sum_{n=1}^{\infty} \arctan(\sinh n) \cdot \arctan\left(\frac{\sinh 1}{\cosh n}\right)$$

*converges, and find the sum.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

*Solution.* For non-negative real numbers $a, b$, we have that

$$\begin{aligned}\arctan\left(\frac{\sinh a}{\cosh b}\right) &= \arctan\left(\frac{e^a - e^{-a}}{e^b + e^{-b}}\right) = \arctan\left(\frac{e^{-(b-a)} - e^{-(b+a)}}{1 + e^{-2b}}\right)\\ &= \arctan\left(\frac{x-y}{1+xy}\right) = \arctan(x) - \arctan(y)\\ &= \arctan(e^{-(b-a)}) - \arctan(e^{-(b+a)}).\end{aligned}$$

Then

$$\arctan\left(\frac{\sinh 1}{\cosh n}\right) = \arctan(e^{-n+1}) - \arctan(e^{-n-1}),$$

and

$$\arctan(\sinh n) = \arctan\left(\frac{\sinh n}{\cosh 0}\right) = \arctan(e^n) - \arctan(e^{-n}) = \frac{\pi}{2} - 2\arctan(e^{-n})$$

where we used the identity $\arctan(x) + \arctan(1/x) = \pi/2$ for $x > 0$. Hence

$$\sum_{n=1}^{N} \arctan(\sinh n) \cdot \arctan\left(\frac{\sinh 1}{\cosh n}\right) = \frac{\pi}{2} S_N - 2T_N.$$

where

$$\begin{aligned}S_N &= \sum_{n=1}^{N} \left(\arctan(e^{-n+1}) - \arctan(e^{-n-1})\right)\\ &= \sum_{n=0}^{N-1} \arctan(e^{-n}) - \sum_{n=2}^{N+1} \arctan(e^{-n})\\ &= \arctan(1) + \arctan(1/e) - \arctan(e^{-N}) - \arctan(e^{-N-1}) \to \frac{\pi}{4} + \arctan(1/e),\end{aligned}$$

and

$$\begin{aligned}T_N &= \sum_{n=1}^{N} \arctan(e^{-n}) \left(\arctan(e^{-n+1}) - \arctan(e^{-n-1})\right)\\ &= \sum_{n=0}^{N-1} \arctan(e^{-n}) \arctan(e^{-n-1}) - \sum_{n=1}^{N} \arctan(e^{-n}) \arctan(e^{-n-1})\\ &= \arctan(1)\arctan(1/e) - \arctan(e^{-N})\arctan(e^{-N-1}) \to \frac{\pi}{4}\arctan(1/e).\end{aligned}$$

Therefore the desired series is convergent and

$$\sum_{n=1}^{\infty} \arctan(\sinh n) \cdot \arctan\left(\frac{\sinh 1}{\cosh n}\right) = \frac{\pi}{2}\left(\frac{\pi}{4} + \arctan(1/e)\right) - 2\left(\frac{\pi}{4}\arctan(1/e)\right) = \frac{\pi^2}{8}.$$

□