Problem 11954
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Proposed by P. Bracken (USA).

Determine the largest constant c and the smallest constant d such that, for all positive integers n,

$$\frac{1}{n-c} \leq \sum_{k=n}^{\infty} \frac{1}{k^2} \leq \frac{1}{n-d}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. For $x > 0$, let

$$\Psi_1(x) = \sum_{k=0}^{\infty} \frac{1}{(k+x)^2}$$

the so-called *trigamma function*. We show that the function

$$F(x) := x - \frac{1}{\Psi_1(x)}$$

is strictly increasing in $(0, +\infty)$. Then, the required constants are easy to be determined:

$$c = \inf_{n\geq 1} \left(n - \frac{1}{\Psi_1(n)}\right) = 1 - \frac{1}{\Psi_1(1)} = 1 - \frac{6}{\pi^2},$$

and

$$d = \sup_{n\geq 1} \left(n - \frac{1}{\Psi_1(n)}\right) = \lim_{n\to+\infty} \left(n - \frac{1}{\Psi_1(n)}\right) = \frac{1}{2}$$

where in the last step we used the fact that

$$\Psi_1(n) = \sum_{k=n}^{\infty} \frac{1}{k^2} = \frac{1}{n} + \frac{1}{2n^2} + o(1/n^2)$$

which follows from the Euler-Maclaurin formula.
Now, by differentiating F we obtain

$$F'(x) = 1 + \frac{\Psi_1'(x)}{\Psi_1^2(x)} = \frac{\Psi_1^2(x) + \Psi_1'(x)}{\Psi_1^2(x)} \quad \text{where} \quad \Psi_1'(x) = -2\sum_{k=0}^{\infty} \frac{1}{(k+x)^3}.$$

Hence, it suffices to show that for $x > 0$, $G(x) := \Psi_1^2(x) + \Psi_1'(x) > 0$.
Since $\lim_{x\to+\infty} G(x) = 0$, it remains to prove that for $x > 0$, $G(x+1) - G(x) < 0$, that is

$$\begin{aligned}\Psi_1^2(x+1) + \Psi_1'(x+1) - \Psi_1^2(x) - \Psi_1'(x) &= \left(\Psi_1(x) - \frac{1}{x^2}\right)^2 + \left(\Psi_1'(x) + \frac{2}{x^3}\right) - \Psi_1^2(x) - \Psi_1'(x)\\ &= -\frac{2\Psi_1(x)}{x^2} + \frac{1}{x^4} + \frac{2}{x^3} = -\frac{2}{x^2}\left(\Psi_1(x) - \frac{1}{x} - \frac{1}{2x^2}\right) < 0\end{aligned}$$

which holds if and only if $H(x) := \Psi_1(x) - \frac{1}{x} - \frac{1}{2x^2} > 0$.
Since $\lim_{x\to+\infty} H(x) = 0$, we still have to show that for $x > 0$, $H(x+1) - H(x) < 0$, that is

$$\begin{aligned}\Psi_1(x+1) - \frac{1}{x+1} - \frac{1}{2(x+1)^2} - \Psi_1(x) + \frac{1}{x} + \frac{1}{2x^2} &= \left(\Psi_1(x) - \frac{1}{x^2}\right) - \frac{2x+3}{2(x+1)^2} - \Psi_1(x) + \frac{2x+1}{2x^2}\\ &= -\frac{1}{2x^2(x+1)^2} < 0\end{aligned}$$

which is true. $\square$