Problem 11953
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Calculate

$$\int_0^\infty \int_0^\infty \frac{\sin(x)\sin(y)\sin(x+y)}{xy(x+y)}\,dx\,dy.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We have that

$$\begin{aligned}
I &:= \int_0^\infty \int_0^\infty \frac{\sin(x)\sin(y)\sin(x+y)}{xy(x+y)}\,dx\,dy = \lim_{R\to+\infty}\int_{x=0}^R \int_{y=0}^R \frac{\sin(x)\sin(y)\sin(x+y)}{xy(x+y)}\,dx\,dy \\
&= \lim_{R\to+\infty} 2\int_{x=0}^R \left(\int_{y=0}^x \frac{\sin(x)\sin(y)\sin(x+y)}{xy(x+y)}\,dy\right)dx \\
&= \lim_{R\to+\infty} \frac{1}{2}\int_{x=0}^R \left(\int_{y=0}^x \frac{\sin(2x)+\sin(2y)-\sin(2x+2y)}{xy(x+y)}\,dy\right)dx.
\end{aligned}$$

Then, after letting $u=2x$, $v=2y=tu$,

$$\begin{aligned}
I &= \lim_{R\to+\infty}\int_{u=0}^R \left(\int_{v=0}^u \frac{\sin(u)+\sin(v)-\sin(u+v)}{uv(u+v)}\,dv\right)du \\
&= \lim_{R\to+\infty}\int_{u=0}^R \left(\int_{t=0}^1 \frac{\sin(u)+\sin(tu)-\sin(u+tu)}{t(1+t)u^2}\,dt\right)du = \lim_{R\to+\infty}\int_{t=0}^1 \frac{f_R(t)}{t(1+t)}\,dt,
\end{aligned}$$

where

$$\begin{aligned}
f_R(t) &:= \int_{u=0}^R \frac{\sin(u)+\sin(tu)-\sin(u+tu)}{u^2}\,du = \left[-\frac{\sin(u)+\sin(tu)-\sin(u+tu)}{u}\right]_{u=0}^R \\
&\quad + \int_{u=0}^R \frac{\cos(u)-\cos((1+t)u)}{u}\,du + t\int_{u=0}^R \frac{\cos(tu)-\cos((1+t)u)}{u}\,du.
\end{aligned}$$

By using the Frullani's integral,

$$\int_0^{+\infty} \frac{\cos(au)-\cos(bu)}{u}\,dt = \ln(b/a) \qquad \text{for } a,b>0,$$

it follows that

$$\lim_{R\to+\infty} f_R(t) = f(t) := \ln(1+t) + t\ln((1+t)/t) = (1+t)\ln(1+t) - t\ln(t).$$

Note that for any $t\in[0,1]$,

$$|f_R(t)-f(t)| \le \int_R^\infty \frac{3}{u^2}\,du = \frac{3}{R}.$$

Hence, by the Dominated Convergence Theorem, we finally obtain

$$\begin{aligned}
I &= \int_0^1 \frac{f(t)}{t(1+t)}\,dt = \int_0^1 \frac{\ln(1+t)}{t}dt - [\ln(t)\ln(1+t)]_0^1 + \int_0^1 \frac{\ln(1+t)}{t}dt \\
&= 2\int_0^1 \frac{\ln(1+t)}{t}dt = 2\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}\int_0^1 t^{n-1}dt = 2\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^2} = \zeta(2) = \frac{\pi^2}{6}.
\end{aligned}$$

□