Problem 11952
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Prove that

$$\sum_{n=1}^{\infty} \frac{2^{2n-1}}{2n+1} \left(\frac{(n-1)!}{(2n-1)!!} \right)^2 = \pi - 2.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We first show by induction that for $N > 0$,

$$S_N := 2\sum_{n=1}^{N-1} \frac{2^{2n-1}}{2n+1} \left(\frac{(n-1)!}{(2n-1)!!} \right)^2 = \sum_{n=1}^{N-1} \frac{4^{2n}}{n^2(2n+1)\binom{2n}{n}^2} = \frac{(2N-1)4^{2N}}{N^2\binom{2N}{N}^2} - 4.$$

The identity is true for $N = 1$. For $N > 1$ we have to verify that

$$\frac{4^{2N}}{N^2(2N+1)\binom{2N}{N}^2} = S_{N+1} - S_N = \left(\frac{(2N+1)4^{2N+2}}{(N+1)^2\binom{2N+2}{N+1}^2} - 4 \right) - \left(\frac{(2N-1)4^{2N}}{N^2\binom{2N}{N}^2} - 4 \right),$$

that is

$$\frac{1}{N^2(2N+1)} = \frac{4}{(2N+1)} - \frac{(2N-1)}{N^2},$$

which holds. Hence

$$\sum_{n=1}^{\infty} \frac{2^{2n-1}}{2n+1} \left(\frac{(n-1)!}{(2n-1)!!} \right)^2 = \frac{1}{2} \lim_{N\to+\infty} S_N = \frac{1}{2} \lim_{N\to+\infty} \frac{(2N-1)4^{2N}}{N^2\binom{2N}{N}^2} - 2 = \pi - 2$$

because by Stirling approximation

$$\lim_{N\to+\infty} \frac{\sqrt{N}}{4^N} \binom{2N}{N} = \frac{1}{\sqrt{\pi}}.$$

□