Problem 11949
(American Mathematical Monthly, Vol.123, December 2016)

Proposed by E. J. Ionascu (USA).

Show that there exists a unique function f from $\mathbb{R}$ to $\mathbb{R}$ such that f is differentiable,

$$2\cos(x+f(x)) - \cos(x) = 1 \quad \text{for all real } x, \text{ and } f(\pi/2) = -\pi/6.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $F(x) := x + f(x)$ then for all $x \in \mathbb{R}$,

$$\cos(F(x)) = \frac{1+\cos(x)}{2} = \cos^2(x/2)$$

that is

$$F(x) = s(x) \cdot \arccos(\cos^2(x/2)) + 2\pi \cdot k(x)$$

where $s(x) \in \{1, -1\}$ and $k(x) \in \mathbb{Z}$.
Since

$$\pm \arccos(\cos^2(x/2)) \in [-\pi/2, \pi/2],$$

it follows that the intervals $[-\pi/2 + 2\pi k, \pi/2 + 2\pi k]$ for $k \in \mathbb{Z}$ are disjoint. Hence, by the continuity of $F(x)$, the function $k(x)$ is identically constant in $\mathbb{R}$. This constant is zero because of the condition $F(\pi/2) = \pi/2 - \pi/6 = \pi/3$.
By a similar reason, since $\arccos(\cos^2(x/2))$ is zero when $x = 2\pi n$ for $n \in \mathbb{Z}$, and it positive otherwise, we have that the sign $s(x)$ is identically constant in each interval $(2\pi n, 2\pi(n+1))$ and $s(x) \equiv 1$ in $(0, 2\pi)$.
Finally, in order to have differentiability, the limits

$$\lim_{x \to 2\pi n^{\pm}} D_x \left(\arccos(\cos^2(x/2))\right) = \pm \frac{1}{\sqrt{2}},$$

imply that sign $s(x)$ changes passing through each point $2\pi n$ and we may conclude that $s(x) = (-1)^{\lfloor \frac{x}{2\pi} \rfloor}$.
Hence there is only one candidate for f, namely

$$f(x) = F(x) - x = (-1)^{\lfloor \frac{x}{2\pi} \rfloor} \cdot \arccos(\cos^2(x/2)) - x.$$

It is easy to verify that f is differentiable, it satisfies the required equation and $f(\pi/2) = -\pi/6$. $\square$