Problem 11947
(American Mathematical Monthly, Vol.123, December 2016)

Proposed by G. Stoica (Canada).

Let n be a positive integer, and let $z_1,\dots,z_n$ be the zeros in $\mathbb{C}$ of z^n+1. For $a>0$, prove

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1}{|z_k-a|^2}=\frac{\sum_{k=0}^{n-1}a^{2k}}{(1+a^n)^2}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. For $z\in\mathbb{C}$ such that $z^n\neq -1$,

$$f(z):=\frac{\sum_{k=0}^{n-1}z^{2k}}{(1+z^n)^2}=\frac{z^{2n}-1}{(z^2-1)(1+z^n)^2}=\frac{z^n-1}{(z^2-1)(1+z^n)}.$$

Let $z_k=\exp(\pi i(2k-1)/n)$ for $k=1,\dots,n$. If $z_k\neq -1$ then

$$\operatorname{Res}(f(z),z_k)=\operatorname{Res}\left(\frac{\frac{z^n-1}{(z^2-1)}}{1+z^n},z_k\right)=\lim_{z\to z_k}\frac{\frac{z^n-1}{z^2-1}}{nz^{n-1}}=\frac{2}{n(z_k-\overline{z_k})}$$

On the other hand, if if n is odd then $z_n=-1$ and

$$\lim_{z\to-1}(z+1)^2f(z)=\lim_{z\to-1}\left(\frac{z^n-1}{z-1}\cdot\frac{z+1}{1+z^n}\right)=\lim_{z\to-1}\frac{1}{nz^{n-1}}=\frac{1}{n},$$

$$\lim_{z\to-1}(z+1)\left(f(z)-\frac{1}{n(z+1)^2}\right)=0.$$

i) If $n=2m$ then $z_k\neq-1$, and by partial fraction decomposition,

$$f(z)=\sum_{k=1}^{n}\frac{A_k}{z-z_k}\quad\text{where}\quad A_k=\lim_{z\to z_k}(z-z_k)f(z)=\operatorname{Res}(f(z),z_k)=\frac{2}{n(z_k-\overline{z_k})}.$$

Since $\overline{z_k}=z_{n+1-k}$, it follows that $A_k=-A_{n+1-k}$ and for $a>0$

$$\begin{aligned}f(a)&=\sum_{k=1}^{m}\frac{A_k}{a-z_k}+\sum_{k=1}^{m}\frac{A_{n+1-k}}{a-z_{n+1-k}}=\sum_{k=1}^{m}A_k\left(\frac{1}{a-z_k}-\frac{1}{a-\overline{z_k}}\right)\\&=\sum_{k=1}^{m}\frac{A_k(z_k-\overline{z_k})}{|z_k-a|^2}=\frac{1}{n}\sum_{k=1}^{n}\frac{1}{|z_k-a|^2}.\end{aligned}$$

ii) If $n=2m+1$ then $z_k\neq-1$ for $k\neq m+1$, and $z_{m+1}=-1$.
Again by partial fraction decomposition,

$$f(z)=\frac{1}{n(z+1)^2}+\sum_{k=1}^{m}\frac{A_k}{z-z_k}+\sum_{k=m+2}^{n}\frac{A_k}{z-z_k}\quad\text{where}\quad A_k=\frac{2}{n(z_k-\overline{z_k})}.$$

Since $\overline{z_k}=z_{n+1-k}$, it follows that $A_k=-A_{n+1-k}$ and for $a>0$

$$\begin{aligned}f(a)&=\frac{1}{n(a+1)^2}+\sum_{k=1}^{m}\frac{A_k}{a-z_k}+\sum_{k=1}^{m}\frac{A_{n+1-k}}{a-z_{n+1-k}}=\frac{1}{n(a+1)^2}+\sum_{k=1}^{m}A_k\left(\frac{1}{a-z_k}-\frac{1}{a-\overline{z_k}}\right)\\&=\frac{1}{n(-1-a)^2}+\sum_{k=1}^{m}\frac{A_k(z_k-\overline{z_k})}{|z_k-a|^2}=\frac{1}{n}\sum_{k=1}^{n}\frac{1}{|z_k-a|^2}.\end{aligned}$$

and we are done. □