Problem 11946
(American Mathematical Monthly, Vol.123, December 2016)

Proposed by M. Omarjee (France).

Let f be a twice differentiable function from $[0,1]$ to $\mathbb{R}$ with f'' continuous on $[0,1]$ and $\int_{1/3}^{2/3} f(x)dx = 0$. Prove

$$4860 \left(\int_0^1 f(x)dx \right)^2 \leq 11 \int_0^1 (f''(x))^2 dx.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $g(x)$ be the piecewise differentiable function defined as

$$g(x) := \begin{cases} x & \text{if } x \in [0, 1/3), \\ 1-2x & \text{if } x \in [1/3, 2/3), \\ x-1 & \text{if } x \in [2/3, 1), \end{cases}$$

and let

$$G(x) := \int_0^x g(t)dt = \begin{cases} \frac{x^2}{2} & \text{if } x \in [0, 1/3), \\ -x^2 + x - \frac{1}{6} & \text{if } x \in [1/3, 2/3), \\ \frac{x^2}{2} - x + \frac{1}{2} & \text{if } x \in [2/3, 1). \end{cases}$$

Since $\int_{1/3}^{2/3} f(x)\,dx = 0$, it follows that

$$\int_0^1 f(x)\,dx = \int_0^{1/3} f(x)\,dx - 2\int_{1/3}^{2/3} f(x)\,dx + \int_{2/3}^1 f(x)\,dx = \int_0^1 f(x)g'(x)\,dx.$$

Hence, by integrating by parts, we obtain

$$\begin{aligned} \int_0^1 f(x)\,dx &= \int_0^1 f(x)\,d(g(x)) = [f(x)g(x)]_0^1 - \int_0^1 g(x)\,d(f(x)) = -\int_0^1 g(x)f'(x)\,dx \\ &= -\int_0^1 f'(x)\,d(G(x)) = -\left[f'(x)G(x)\right]_0^1 + \int_0^1 f''(x)G(x)dx = \int_0^1 f''(x)G(x)dx. \end{aligned}$$

Finally, by Cauchy-Schwarz inequality, we get

$$\left(\int_0^1 f(x)\,dx \right)^2 \leq \int_0^1 (G(x))^2\,dx \cdot \int_0^1 (f''(x))^2\,dx = \frac{11}{4860} \int_0^1 (f''(x))^2\,dx$$

because

$$\int_0^1 (G(x))^2\,dx = 2\int_0^{1/3} \frac{x^4}{4}\,dx + 2\int_{1/3}^{1/2} \left(-x^2 + x - \frac{1}{6}\right)^2 dx = \frac{11}{4860}.$$

□