Problem 11945
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Proposed by M. Lukarevski (Macedonia).

Let a, b, and c be the lengths of the sides of triangle ABC opposite A, B, and C, respectively, and let w_a , w_b , w_c be the lengths of the corresponding angle bisectors. Prove

$$\frac{a}{w_a}+\frac{b}{w_b}+\frac{c}{w_c}\geq 2\sqrt{3}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. By using the inequality $x+y\geq 2\sqrt{xy}$ twice, we have that

$$w_a=\frac{2bc}{b+c}\cdot\cos(A/2)=\sqrt{bc}\cdot\cos(A/2)=\sqrt{s(s-a)}=\frac{2\sqrt{s(3s-3a)}}{2\sqrt{3}}\leq\frac{4s-3a}{2\sqrt{3}}$$

where $s=(a+b+c)/2$. Hence

$$\frac{a}{w_a}+\frac{b}{w_b}+\frac{c}{w_c}\geq f(a)+f(b)+f(c)\geq 3f\left(\frac{a+b+c}{3}\right)=3f\left(\frac{2s}{3}\right)=2\sqrt{3}$$

where we used the fact that $f(x)=\frac{2\sqrt{3}x}{4s-3x}$ is convex in $[0,4s/3)$ (note that $3\max(a,b,c)<2(a+b+c)=4s$). $\square$