Problem 11942
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Proposed by F. Parvanescu (Romania).

In acute triangle ABC, let D be the foot of the altitude from A, let E be the foot of the perpendicular from D to AC, and let F be a point on segment DE. Prove that AF is perpendicular to BE if and only if $|FE|/|FD| = |BD|/|CD|$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We may assume without loss of generality that

$$A = (0, a),\ B = (-b, 0),\ C = (c, 0),\ D = (0, 0)$$

where $a, c > 0$, $b \geq 0$, and $a^2 + c^2 = 1$. Then, it is easy to see that

$$E = (|DC| \sin^2 C, |DC| \sin C \cos C) = (a^2 c, ac^2).$$

Let $F = tE$ with $t \in (0, 1]$. Hence

$$\frac{|FE|}{|FD|} = \frac{1-t}{t}, \quad \frac{|BD|}{|CD|} = \frac{b}{c}$$

and $|FE|/|FD| = |BD|/|CD|$ iff

$$t = \frac{c}{c+b}.$$

Moroever, AF is perpendicular to BE if and only the scalar product of the vectors AF and BE is zero:

$$0 = AF \cdot BE = (tE - A) \cdot (E - B) = (ta^2 c, tac^2 - a) \cdot (a^2 c + b, ac^2) = ta^4 c^2 + ta^2 cb + ta^2 c^4 - a^2 c^2,$$

which holds iff

$$t = \frac{a^2 c^2}{a^4 c^2 + a^2 cb + a^2 c^4} = \frac{c}{c+b}$$

and we are done. $\square$