Problem 11941
(American Mathematical Monthly, Vol.123, November 2016)

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Let

$$L = \lim_{n\to\infty} \int_0^1 \sqrt[n]{x^n + (1-x)^n}\, dx.$$

(a) Find L.
(b) Find

$$\lim_{n\to\infty} n^2 \left(\int_0^1 \sqrt[n]{x^n + (1-x)^n}\, dx - L \right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution.
(a) Let $t = \frac{x}{1-x}$, then $x = \frac{t}{1+t}$, $dx = \frac{dt}{(1+t)^2}$, and

$$\int_0^1 \sqrt[n]{x^n + (1-x)^n}\, dx = 2\int_0^{1/2} (1-x) \sqrt[n]{1 + \left(\frac{x}{1-x}\right)^n}\, dx = 2\int_0^1 \frac{\sqrt[n]{1+t^n}}{(1+t)^3}\, dt.$$

Since for all $t \in [0,1]$,

$$\frac{\sqrt[n]{1+t^n}}{(1+t)^3} \to \frac{1}{(1+t)^3} \quad \text{and} \quad 0 \leq \frac{\sqrt[n]{1+t^n}}{(1+t)^3} \leq \frac{2}{(1+t)^3},$$

by the Lebesgue's Dominated Convergence Theorem,

$$L = \lim_{n\to\infty} \int_0^1 \sqrt[n]{x^n + (1-x)^n}\, dx = 2\int_0^1 \frac{1}{(1+t)^3}\, dt = \left[-\frac{1}{(1+t)^2} \right]_0^1 = \frac{3}{4}.$$

(b) By letting $s = t^n$, we obtain $t = s^{1/n}$, $dt = \frac{1}{n} s^{1/n-1} ds$, and

$$\begin{aligned} n^2 \left(\int_0^1 \sqrt[n]{x^n + (1-x)^n}\, dx - L \right) &= 2n^2 \int_0^1 \frac{\sqrt[n]{1+t^n} - 1}{(1+t)^3}\, dt \\ &= 2n \int_0^1 \frac{(1+s)^{1/n} - 1}{(1+s^{1/n})^3}\, s^{1/n-1} ds \\ &= 2n \int_0^1 \frac{\exp(\frac{1}{n}\ln(1+s)) - \exp(0)}{(1+s^{1/n})^3}\, s^{1/n-1} ds \\ &= 2\int_0^1 \frac{\exp(h_n(s))\ln(1+s)}{(1+s^{1/n})^3}\, s^{1/n-1} ds \end{aligned}$$

where $0 < h_n(s) < \frac{1}{n}\ln(1+s)$ is given by the Mean Value Theorem. Now for all $s \in (0,1]$,

$$\frac{\exp(h_n(s))\ln(1+s)}{(1+s^{1/n})^3}\, s^{1/n-1} \to \frac{\ln(1+s)}{8s} \quad \text{and} \quad 0 \leq \frac{\exp(h_n(s))\ln(1+s)}{(1+s^{1/n})^3}\, s^{1/n-1} \leq \frac{2\ln(1+s)}{s}.$$

Hence, by the Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned} \lim_{n\to\infty} n^2 \left(\int_0^1 \sqrt[n]{x^n + (1-x)^n}\, dx - L \right) &= \frac{1}{4}\int_0^1 \frac{\ln(1+s)}{s}\, ds = \frac{1}{4}\int_0^1 \sum_{k=1}^{\infty} \frac{(-1)^{k-1} s^{k-1}}{k} \\ &= \frac{1}{4}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} = \frac{\pi^2}{48}. \end{aligned}$$

□