Problem 11940
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Proposed by H. Ohtsuka (Japan).

Let $T_n = n(n+1)/2$ *and* $C(n,k) = (n-2k)\binom{n}{k}$*. For* $n \geq 1$*, prove*

$$\sum_{k=0}^{n-1} C(T_n,k)C(T_{n+1},k) = \frac{n^3-2n^2+4n}{n+2}\binom{T_n}{n}\binom{T_{n+1}}{n}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. It suffices to show that for $n, m \geq 1$,

$$\sum_{k=0}^{m-1} C(T_n,k)C(T_{n+1},k) = F(n,m) := \frac{m^2(n(n+2)+4-4m)}{n(n+2)}\binom{T_n}{m}\binom{T_{n+1}}{m}.$$

We proceed by induction with respect to m.
Basic step. For $m = 1$,

$$\sum_{k=0}^{0} C(T_n,k)C(T_{n+1},k) = T_n \cdot T_{n+1} = F(n,1).$$

Inductive step. For $m > 1$, we have to prove that

$$C(T_n,m)C(T_{n+1},m) = F(n,m+1) - F(n,m)$$

that is

$$n(n+2)(T_n-2m)(T_{n+1}-2m) = (n(n+2)-4m)(T_n-m)(T_{n+1}-m) - m^2(n(n+2)+4-4m).$$

which can be verified straightforwardly. □