Problem 11937
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Proposed by J. C. Sampedro (Spain).

Let s be a complex number such that $\mathrm{Re}(s) > 0$. *Prove*

$$\int_0^1 \int_0^1 \frac{(xy)^{s-1} - y}{(1-xy)\log(xy)}\,dxdy = \frac{\Gamma'(s)}{\Gamma(s)}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We show a more general result:

Let s be a complex number such that $\mathrm{Re}(s) > 0$ and let n be a positive integer. Then

$$\int_0^1 \int_0^1 \frac{(xy)^{s-1} - y^n}{(1-xy)\log(xy)}\,dxdy = \frac{\Gamma'(s)}{\Gamma(s)} - \frac{\log(n!)}{n}.$$

Let $u = xy$, then $x = u/y$, $dx = du/y$ and

$$\begin{aligned}
\int_0^1 \int_0^1 \frac{(xy)^{s-1} - y^n}{(1-xy)\log(xy)}\,dxdy &= \int_{u=0}^1 \int_{y=u}^1 \frac{u^{s-1} - y^n}{(1-u)\log(u)}\,\frac{dudy}{y} \\
&= \int_{u=0}^1 \frac{1}{(1-u)\log(u)} \left(\int_{y=u}^1 \left(\frac{u^{s-1}}{y} - y^{n-1} \right) dy \right) du \\
&= \int_{u=0}^1 \frac{1}{(1-u)\log(u)} \left[u^{s-1}\log(y) - \frac{y^n}{n} \right]_{y=u}^1 du \\
&= -\int_0^1 \left(\frac{u^{s-1}}{(1-u)} + \frac{1}{\log(u)} \right) du + \int_0^1 \left(\frac{1-u^n}{n(1-u)} - 1 \right) \frac{du}{\log(u)} \\
&= -\int_0^1 \left(\frac{u^{s-1}}{(1-u)} + \frac{1}{\log(u)} \right) du - \frac{1}{n}\sum_{k=1}^{n-1} \int_0^1 \frac{u^k - 1}{\log(u)}\,du \\
&= \frac{\Gamma'(s)}{\Gamma(s)} - \frac{\log(n!)}{n},
\end{aligned}$$

where in the last step we used a known integral representation of the digamma function (see 4.281.4 in *Table of integrals series and products* by Gradshteyn and Ryzhik),

$$\int_0^1 \left(\frac{u^{s-1}}{(1-u)} + \frac{1}{\log(u)} \right) du = -\frac{\Gamma'(s)}{\Gamma(s)} \qquad \text{for } \mathrm{Re}(s) > 0$$

and the fact that for $t \geq 0$,

$$f(t) := \int_0^1 \frac{u^t - 1}{\log(u)}\,du = \log(t+1)$$

which can be easily obtained by noting that $f(0) = 0$ and

$$f'(t) = \int_0^1 u^t\,du = \frac{1}{t+1}.$$

□