Problem 11935
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Proposed by D. M. Bătineu-Giurgiu (Romania), Anastasios Kotronis (Greece), and Neculai Stanciu (Romania).

Let f be a function from $\mathbb{Z}^+$ to $\mathbb{R}^+$ such that $\lim_{n\to\infty} f(n)/n = a$, where $a > 0$. Find

$$\lim_{n\to\infty}\left(\sqrt[n+1]{\prod_{k=1}^{n+1} f(k)} - \sqrt[n]{\prod_{k=1}^{n} f(k)}\right).$$

Solution proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France, and Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", Italy.

Solution. Let

$$a_n = \sqrt[n]{\prod_{k=1}^{n} f(k)}$$

and let $x_n = (a_n/n)^n$. Then by Stolz-Cesaro,

$$\begin{aligned}
\lim_{n\to\infty} \ln\left(\frac{a_n}{n}\right) &= \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \ln f(k) - n\ln n}{n} \\
&\overset{SC}{=} \lim_{n\to\infty} \left(\ln(f(n+1)) - (n+1)\ln(n+1) + n\ln n\right) \\
&= \lim_{n\to\infty} \left(\ln\left(\frac{f(n+1)}{n+1}\right) - n\ln\left(1+\frac{1}{n}\right)\right) = \ln\left(\frac{a}{e}\right).
\end{aligned}$$

Hence $\frac{a_n}{n} = \sqrt[n]{x_n} \to \frac{a}{e}$, which implies that $\frac{x_{n+1}}{x_n} \to \frac{a}{e}$. Moreover

$$\frac{a_{n+1}}{a_n} \to 1, \quad \text{and} \quad \left(\frac{a_{n+1}}{a_n}\right)^n = \frac{(n+1)^n x_{n+1}^{\frac{n}{n+1}}}{n x_n} = \left(1+\frac{1}{n}\right)^n \cdot \frac{x_{n+1}}{x_n} \cdot \frac{1}{\sqrt[n+1]{x_{n+1}}} \to e.$$

Therefore

$$\sqrt[n+1]{\prod_{k=1}^{n+1} f(k)} - \sqrt[n]{\prod_{k=1}^{n} f(k)} = a_{n+1} - a_n = \frac{a_n}{n} \cdot \frac{\left(\frac{a_{n+1}}{a_n} - 1\right)}{\ln\left(1+\left(\frac{a_{n+1}}{a_n}-1\right)\right)} \cdot \ln\left(\left(\frac{a_{n+1}}{a_n}\right)^n\right) \to \frac{a}{e}.$$

□