Problem 11932
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Proposed by H. Ohtsuka (Japan).

Let r be an integer. Prove that

$$\sum_{n=-\infty}^{\infty} \arctan\left(\frac{\sinh r}{\cosh n}\right) = \pi r.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We have that

$$\begin{aligned}\arctan\left(\frac{\sinh r}{\cosh n}\right) &= \arctan\left(\frac{e^r - e^{-r}}{e^n + e^{-n}}\right) = \arctan\left(\frac{e^{-(n-r)} - e^{-(n+r)}}{1 + e^{-2n}}\right)\\ &= \arctan\left(\frac{x-y}{1+xy}\right) = \arctan(x) - \arctan(y)\\ &= \arctan(e^{-(n-r)}) - \arctan(e^{-(n+r)})\end{aligned}$$

because $xy = e^{-2n} > -1$.
Without loss of generality, we may assume that $r \geq 0$ (otherwise we use $\sinh(r) = -\sinh(-r)$).
Hence, since $\cosh(n) = \cosh(-n)$, we have that

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \arctan\left(\frac{\sinh r}{\cosh n}\right) &= 2\sum_{n=1}^{\infty} \arctan\left(\frac{\sinh r}{\cosh n}\right) + \arctan(\sinh r)\\ &= 2\sum_{n=1}^{\infty} \left[\arctan(e^{-(n-r)}) - \arctan(e^{-(n+r)})\right] + \arctan(e^r) - \arctan(e^{-r}).\end{aligned}$$

Now $\sum_{n=1}^{\infty} \arctan(e^{-(n\pm r)})$ is convergent and we can split the series,

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \arctan\left(\frac{\sinh r}{\cosh n}\right) &= 2\sum_{n=1}^{\infty} \arctan(e^{-(n-r)}) - 2\sum_{n=1}^{\infty} \arctan(e^{-(n+r)}) + \arctan(e^r) - \arctan(e^{-r})\\ &= 2\sum_{m\geq 1-r} \arctan(e^{-m}) - 2\sum_{m\geq r+1}^{\infty} \arctan(e^{-m}) + \arctan(e^r) - \arctan(e^{-r})\\ &= 2\sum_{1-r\leq m\leq r} \arctan(e^{-m}) + \arctan(e^r) - \arctan(e^{-r})\\ &= 2\sum_{-r\leq m\leq r} \arctan(e^{-m}) - \arctan(e^r) - \arctan(e^{-r})\\ &= 2\sum_{1\leq m\leq r} \left[\arctan(e^m) + \arctan(e^{-m})\right] + 2\arctan(1) - \arctan(e^r) - \arctan(e^{-r})\\ &= 2\sum_{1\leq m\leq r} \frac{\pi}{2} + 2\cdot\frac{\pi}{4} - \frac{\pi}{2} = \pi r\end{aligned}$$

where, at the end, we used the identity $\arctan(x) + \arctan(1/x) = \pi/2$ for $x > 0$. □