Problem 11930
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Find

$$\sum_{n=1}^{\infty} \sinh^{-1}\left(\frac{1}{\sqrt{2^{n+2}+2}+\sqrt{2^{n+1}+2}}\right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let

$$a_n = \frac{1}{\sqrt{2^{n+2}+2}+\sqrt{2^{n+1}+2}} \quad \text{and} \quad b_n = \frac{\sqrt{2^n+1}-\sqrt{3}}{2^{\frac{n+1}{2}}}.$$

Then, it can be easily verified that

$$b_{n+1}\sqrt{1+b_n^2} - b_n\sqrt{1+b_{n+1}^2} = a_n.$$

By the known identity,

$$\sinh^{-1}(x\sqrt{1+y^2} - y\sqrt{1+x^2}) = \sinh^{-1}(x) - \sinh^{-1}(y),$$

it follows that

$$\sum_{n=1}^{N} \sinh^{-1}(a_n) = \sum_{n=1}^{N} \left(\sinh^{-1}(b_{n+1}) - \sinh^{-1}(b_n)\right) = \sinh^{-1}(b_{N+1}) - \sinh^{-1}(b_1).$$

Now $b_1 = 0$, $b_{N+1} \to 1/\sqrt{2}$, and therefore

$$\sum_{n=1}^{\infty} \sinh^{-1}\left(\frac{1}{\sqrt{2^{n+2}+2}+\sqrt{2^{n+1}+2}}\right) = \lim_{N\to\infty} \sinh^{-1}(b_{N+1}) = \sinh^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\ln(2+\sqrt{3})}{2}.$$

$\square$