Problem 11928
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Proposed by H. Ohtsuka (Japan).

For positive integers n *and* m *and for a sequence* $\{a_i\}_{i\geq 1}$, *prove*

$$\sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j}a_{i+j}=\sum_{k=0}^{n+m}\binom{n+m}{k}a_k$$

and

$$\sum_{0\leq i<j\leq n}\binom{n}{i}\binom{n}{j}\binom{i+j}{n}=\sum_{0\leq i<j\leq n}\binom{n}{i}\binom{n}{j}^2.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. The first identity is a simple application of Vandermonde's identity,

$$[z^{n+m}](z+1)^n\cdot(z+1)^m\cdot f(z)=\sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{n-i}\binom{m}{m-j}a_{i+j}=\sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j}a_{i+j}$$

and

$$[z^{n+m}](z+1)^{n+m}\cdot f(z)=\sum_{k=0}^{n+m}\binom{n+m}{n+m-k}a_k=\sum_{k=0}^{n+m}\binom{n+m}{k}a_k$$

where $f(z):=\sum_{k\geq 0}a_k z^k$.
As regards the second identity, we have to show that $L_n=R_n$ where

$$L_n:=\sum_{0\leq i<j\leq n}\binom{n}{i}\binom{n}{j}\binom{i+j}{n}=\sum_{0\leq j<i\leq n}\binom{n}{i}\binom{n}{j}\binom{i+j}{n},$$

and

$$R_n:=\sum_{0\leq i<j\leq n}\binom{n}{i}\binom{n}{j}^2=\sum_{0\leq j<i\leq n}\binom{n}{n-i}\binom{n}{n-j}^2=\sum_{0\leq j<i\leq n}\binom{n}{i}\binom{n}{j}^2.$$

Let $m=n$ and $a_k=\binom{k}{n}$ in the first one, then

$$\sum_{i=0}^{n}\binom{n}{i}^2\binom{2i}{n}+2L_n=\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n}{i}\binom{n}{j}\binom{i+j}{n}=\sum_{k=0}^{2n}\binom{2n}{k}\binom{k}{n}=\binom{2n}{n}\sum_{k=n}^{2n}\binom{n}{k-n}=2^n\cdot\binom{2n}{n}.$$

Moreover

$$\sum_{i=0}^{n}\binom{n}{i}^3+2R_n=\left(\sum_{i=0}^{n}\binom{n}{i}\right)\cdot\left(\sum_{j=0}^{n}\binom{n}{j}^2\right)=2^n\cdot\binom{2n}{n}.$$

Therefore, it suffices to prove that

$$\sum_{i=0}^{n}\binom{n}{i}^3=\sum_{i=0}^{n}\binom{n}{i}^2\binom{2i}{n}$$

which is known as Strehl's identity. For the sake of completeness, we give a short proof below:

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}^3&=\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i}\sum_{k=0}^{i}\binom{i}{i-k}\binom{n-i}{k}=\sum_{k=0}^{n}\binom{n}{k}^2\sum_{i=k}^{n}\binom{n-k}{i}\binom{n-k}{n-i}\\&=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2(n-k)}{n}=\sum_{k=0}^{n}\binom{n}{n-k}^2\binom{2k}{n}=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}.\end{aligned}$$

□