Problem 11926
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Let k be an integer, $k \geq 2$. Find

$$I_k := \int_0^{+\infty} \frac{\ln|1-x|}{x^{1+1/k}}\, dx.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We have that

$$\begin{aligned}\int_1^{+\infty} \frac{\ln|1-x|}{x^{1+1/k}}\, dx &= \int_1^{+\infty} \frac{\ln(x-1)}{x^{1+1/k}}\, dx = \int_1^{0} \frac{\ln(1/t-1)}{(1/t)^{1+1/k}}\, d(1/t)\\ &= \int_0^1 \frac{\ln(1-t)}{t^{1-1/k}}\, dt - \int_0^1 \frac{\ln t}{t^{1-1/k}}\, dt.\end{aligned}$$

Now

$$\int_0^1 \frac{\ln t}{t^{1-1/k}}\, dt = \left[-k^2 t^{1/k} + k t^{1/k} \ln t\right]_{t=0^+}^1 = -k^2.$$

Moreover if $\alpha \in (0,2)$ then

$$-\int_0^1 \frac{\ln(1-t)}{t^\alpha}\, dt = \int_0^1 \frac{\sum_{j\geq 1} \frac{t^j}{j}}{t^\alpha}\, dt = \sum_{j\geq 1} \frac{1}{j} \int_0^1 t^{j-\alpha}\, dt = \sum_{j\geq 1} \frac{1}{j(j+1-\alpha)},$$

where we can swap the sum with the integral by monotone convergence theorem: for $t \in (0,1)$ and for any $N \geq 1$,

$$0 \leq \sum_{j=1}^N \frac{t^j}{j} \leq -\ln(1-t)$$

and $t^{-\alpha} \sum_{j\geq 1} \frac{t^j}{j} = -\frac{\ln(1-t)}{t^\alpha}$ is integrable on $(0,1)$. Hence

$$\begin{aligned}I_k &= k^2 + \int_0^1 \frac{\ln(1-t)}{t^{1-1/k}}\, dt + \int_0^1 \frac{\ln(1-t)}{t^{1+1/k}}\, dt\\ &= k^2 - \sum_{j\geq 1} \frac{1}{j(j+\frac{1}{k})} - \sum_{j\geq 1} \frac{1}{j(j-\frac{1}{k})}\\ &= k^2 + 2\sum_{j\geq 1} \frac{1}{\frac{1}{k^2} - j^2} = k\pi \cot(\pi/k)\end{aligned}$$

where in the last step we used the known identity

$$\pi \cot(\pi z) = \frac{1}{z} + 2z \sum_{j\geq 1} \frac{1}{z^2 - j^2}$$

which holds for $z \notin \mathbb{Z}$. $\square$