Problem 11924
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Proposed by C. I. Vălean (Romania).

Calculate

$$\int_0^{\pi/2} \frac{\{\tan x\}}{\tan x}\, dx.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Let $t_n = \arctan(n)$, then

$$I := \int_0^{\pi/2} \frac{\{\tan x\}}{\tan x}\, dx = \int_0^{\pi/2} \left(1 - \frac{\lfloor \tan x \rfloor}{\tan x}\right)\, dx = \frac{\pi}{2} - \sum_{n=1}^{\infty} n \int_{t_n}^{t_{n+1}} \frac{dx}{\tan x}$$
$$= \frac{\pi}{2} - \sum_{n=1}^{\infty} n \left(\ln(\sin t_{n+1}) - \ln(\sin t_n)\right).$$

Since $\sin(\arctan t) = \frac{1}{\sqrt{1+1/t^2}}$ when $t > 0$, it follows that for $N \geq 2$,

$$\sum_{n=1}^{N-1} n \left(\ln(\sin t_{n+1}) - \ln(\sin t_n)\right) = \sum_{n=1}^{N-1} \left[(n+1)\ln(\sin t_{n+1}) - n\ln(\sin t_n)\right] - \sum_{n=1}^{N-1} \ln(\sin t_{n+1})$$
$$= N \ln(\sin t_N) - \sum_{n=0}^{N-1} \ln(\sin t_{n+1})$$
$$= N \ln\left(\frac{1}{\sqrt{1+1/N^2}}\right) - \ln\left(\prod_{n=1}^{N} \frac{1}{\sqrt{1+1/n^2}}\right).$$

Hence, by taking the limit as $N \to +\infty$, we obtain

$$I = \frac{\pi}{2} - \frac{1}{2}\ln\left(\prod_{n=1}^{\infty}\left(1 + \frac{1}{n^2}\right)\right) = \frac{\pi}{2} - \frac{1}{2}\ln\left(\frac{\sin(\pi i)}{\pi i}\right) = \frac{\pi}{2} - \frac{1}{2}\ln\left(\frac{\sinh(\pi)}{\pi}\right) = \frac{1}{2}\ln\left(\frac{2\pi}{1 - e^{-2\pi}}\right),$$

where we used the sine product formula

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right).$$

□